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Semi-analytical solution for flow in leaky unconfined aquifer—aquitard systems

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Summary This study presents a semi-analytical solution for the problem of leakage in an unconfined aquifer bounded below by an aquitard of finite or semi-infinite extent. The homogeneous anisotropic unconfined aquifer of infinite radial extent is pumped continuously at a constant rate from a well of infinitesimal radius. The aquitard is also homogeneous, anisotropic and of infinite radial extent. Flow in both the aquifer and the aquitard is allowed to occur both vertically and horizontally. Exact solutions to the governing equations given in this work are developed in the double Laplace—Hankel transform space for drawdown response in the unconfined aquifer and the underlying aquitard. The inverse transforms of the solutions are obtained numerically. The theoretical results show that leakage can cause significant departure, at both early and late times, from the solution with no leakage. The solution presented here can be used in least-squares routines for estimation of hydraulic parameters for two-layered unconfined aquifer—aquitard systems.

Introduction

It has long been recognized that leakage strongly affects the drawdown response of confined aquifers that are pumped continuously at a constant rate and are bounded by aquitards. The first major attempt to mathematically model the effect of leakage in confined aquifer flow was made by Hantush and Jacob (1955) who presented what has come to be referred to as the classical theory of leakage. To obtain their solution Hantush and Jacob (1955) assumed that the confined aquifer was bounded from below and above by aquitards of vertical finite extent in which flow was entirely vertical and the effect of elastic storage was negligible. The assumption of negligible effect of the elastic storage of the aquitards led to a steady-state aquitard flow problem that yielded a linear distribution of hydraulic head in the aquitards. For the confined aquifer Hantush and Jacob (1955) assumed that flow was essentially horizontal. A major limitation of the classical leakage theory is the assumption that the storage of the aquitards can be neglected.

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Nomenclature

$K_{r,1}$	aquifer horizontal hydraulic conductivity (LT^{-1})	b_1	aquifer thickness (L)
$K_{z,1}$	aquifer vertical hydraulic conductivity (LT^{-1})	b_2	aquitard thickness (L)
$S_{s,1}$	aquifer specific storage (L^{-1})	z	observation point vertical coordinate (L)
S_Y	aquifer specific yield (−)	r	observation point radial coordinate (L)
$K_{r,2}$	aquitard horizontal hydraulic conductivity (LT^{-1})	z_1	observation well top of screen (L)
$K_{z,2}$	aquitard vertical hydraulic conductivity (LT^{-1})	z_2	observation well bottom of screen (L)
$S_{s,2}$	aquitard specific storage (L^{-1})	s_1	drawdown in aquifer (L)
$\alpha_{r,1}$	aquifer horizontal hydraulic diffusivity (L^2T^{-1})	s_2	drawdown in aquitard (L)
$\alpha_{z,1}$	aquifer vertical hydraulic diffusivity (L^2T^{-1})	Q	pumping well discharge rate (L^3T^{-1})
$\alpha_{r,2}$	aquitard horizontal hydraulic diffusivity (L^2T^{-1})	p	Laplace transform parameter
$\alpha_{z,2}$	aquitard vertical hydraulic diffusivity (L^2T^{-1})	a	Hankel transform parameter

Subsequently, Hantush (1960) presented the modified leakage theory in which aquitard elastic storage was taken into account. As in classical leakage theory, flow in the aquitards was assumed to be entirely vertical, whereas that in the aquifer was assumed to be horizontal. Hantush (1960) presented analytical solutions only for early and late time. A more complete analytical solution to the problem, which is not restricted to early and late time behavior, was developed by Neuman and Witherspoon (1969a,b). They considered two confined aquifers, in which flow was assumed to be entirely horizontal, separated by an aquitard in which flow was assumed to be entirely vertical. To justify the assumption of vertical flow in aquitards and horizontal flow in the aquifer, Neuman and Witherspoon (1969b) stated that ''when the permeabilities of the aquifers are two or more orders of magnitude greater than that of the aquitard, errors introduced by this assumption are usually less than 5%.''

The effect of leakage on flow in an unconfined aquifer was first considered by Ehlig and Halepaska (1976) in their numerical (finite difference) solution of a coupled confined-unconfined aquifer problem. They adopted the Boulton (1954) model to simulate unconfined aquifer flow and the Hantush and Jacob (1955) model to simulate leakage through the common boundary of the system. No analytical solution was developed. Zlotnik and Zhan (2005) developed an analytical solution for the problem of flow in a coupled unconfined aquifer–aquitard system where the horizontal flow component in the aquitard is neglected. Zhan and Bian (2006) extended the work of Zlotnik and Zhan (2005) and developed analytical and semi-analytical methods for computing the leakage rate and volume induced by pumping based on the works of Hantush and Jacob (1955) and Butler and Tsou (2003). Zhan and Bian (2006) also neglect horizontal flow in the aquitard. The assumption of strictly vertical flow in the aquitard is based on the work of Neuman and Witherspoon (1969b) as discussed above. Additionally, Zlotnik and Zhan (2005) and Zhan and Bian (2006) restrict their solutions to the case of an aquitard of semi-infinite vertical extent. In this work, we develop a more general solution with respect to permissible values of aquitard hydraulic conductivity and aquitard vertical extent.

The purpose of this work is to present an exact analytical solution (in the double Laplace–Hankel transform space) to the governing equations presented herein for the problem of flow with leakage in an unconfined aquifer bounded from below by an aquitard of finite or semi-infinite vertical extent. The analytical solution obtained in the double Laplace–Hankel transform space is inverted numerically. It is developed for the case of a fully penetrating well of infinitesimal radius pumping continuously at a constant rate from an aquifer of infinite radial extent. Water release due to water table decline is simulated in the manner of Neuman (1972); that is, flow in the unsaturated zone above the water table is assumed to have negligible effect on flow in the saturated zone. Cooley (1971) solved the saturated–unsaturated flow problem numerically, and recently, Tartakovsky and Neuman (2007) have developed an analytical solution to this problem. We focus on incorporating leakage into the unconfined aquifer flow problem, and we adopt the water table response developed by Neuman (1972) for simplicity. Moench (1994) demonstrated that when the parameter estimation procedure is done correctly, it is possible to obtain reasonable estimates of specific storage when water table response is modeled in the manner of Neuman (1972).

The solution developed here is for a two layered unconfined aquifer–aquitard system. The solution for a two-layered system can be used as an approximation of a multilayered system where layers below the unconfined aquifer are combined into a single layer bounded below by an impermeable boundary. We do not adopt the assumptions of horizontal flow in the aquifer and vertical flow in the aquitard. Coupling between unconfined aquifer and aquitard flow is accomplished through continuity conditions (of head and normal flux) imposed at their common boundary. Additionally, the unconfined aquifer and the aquitard are considered to be anisotropic. Theoretical results compare well to numerical results computed with MODFLOW. They also show that leakage can lead to significant deviation of the unconfined aquifer response from the standard response predicted by the Neuman (1972) solution.

Governing flow equations

To determine the drawdown response, s_1, of an unconfined aquifer from which water is pumped continuously at a constant rate, Q, through a fully penetrating well of infinitesimal radius, we begin by defining the governing flow equation as

$$S_{s,1}\frac{\partial s_1}{\partial t}=\frac{K_{r,1}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s_1}{\partial r}\right)+K_{z,1}\frac{\partial^2 s_1}{\partial z^2}, \quad (1)$$

where $S_{s,1}$ is the specific storage, $K_{r,1}$ and $K_{z,1}$ are the radial and vertical hydraulic conductivities, respectively, and (r,z,t) are space–time coordinates (see Fig. 1). It is assumed here that space coordinates are oriented parallel to the respective principal directions of hydraulic conductivity. The aquifer flow equation (1) is solved subject to the following initial and boundary conditions:

$$s_1(r,z,0)=0, \quad (2)$$

$$\lim_{r\to\infty} s_1(r,z,t)=0, \quad (3)$$

$$\lim_{r\to 0}\int_0^{b_1} r\frac{\partial s_1}{\partial r}dz=-\frac{Q}{2\pi K_{r,1}}, \quad (4)$$

$$-K_{z,1}\left.\frac{\partial s_1}{\partial z}\right|_{z=b_1}=S_Y\left.\frac{\partial s_1}{\partial t}\right|_{z=b_1}, \quad (5)$$

where Q is the volume flow rate from the well. Eq. (5) is the linearized form of the kinematic boundary condition at the water table (Neuman, 1972). It is convenient to rewrite Eq. (1) and the associated initial and boundary conditions (Eqs. (2)–(5)) in dimensionless form as

$$\frac{\partial s_{D,1}}{\partial t_D}=\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial s_{D,1}}{\partial r_D}\right)+\kappa_1\frac{\partial^2 s_{D,1}}{\partial z_D^2}, \quad (6)$$

$$s_{D,1}(r_D,z_D,0)=0, \quad (7)$$

$$\lim_{r_D\to\infty} s_{D,1}(r_D,z_D,t_D)=0, \quad (8)$$

$$\lim_{r_D\to 0}\int_0^1 r_D\frac{\partial s_{D,1}}{\partial r_D}dz_D=-2, \quad (9)$$

$$-\left.\frac{\partial s_{D,1}}{\partial z_D}\right|_{z_D=1}=\frac{1}{\alpha_{D,Y}}\left.\frac{\partial s_{D,1}}{\partial t_D}\right|_{z_D=1}, \quad (10)$$

where $s_{D,1}=s_1/H_c$, $z_D=z/b_1$, $r_D=r/b_1$, $t_D=\alpha_{r,1}t/b_1^2$, $\kappa_1=K_{z,1}/K_{r,1}$ (unconfined aquifer anisotropy ratio), $\alpha_{D,Y}=\sigma\kappa_1$ and $\sigma=b_1S_{s,1}/S_Y$. In the above development we have followed common convention in hydrogeology by defining the characteristic head as $H_c=Q/(4\pi b_1\ K_{r,1})$. A summary of all dimensionless parameters used in this work is given in Table 1.

The drawdown response in the aquitard, s_2, due to pumping in the overlying unconfined aquifer is determined by solving the following flow problem:

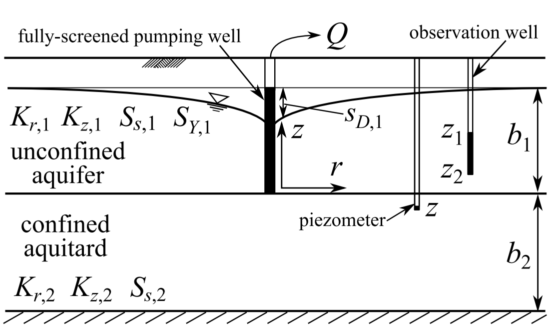

Figure 1 A schematic diagram of an unconfined aquifer–aquitard system.

Table 1 Definitions of dimensionless quantities

Dimensionless parameter	Expression
$\alpha_{D,r}$	$\alpha_{r,2}/\alpha_{r,1}$
$\alpha_{D,z}$	$\alpha_{z,2}/\alpha_{z,1}$
$K_{D,r}$	$K_{r,2}/K_{r,1}$
$K_{D,z}$	$K_{z,2}/K_{z,1}$
$\alpha_{D,Y}$	$b_1S_{S,1}\ K_{z,1}/(S_YK_{r,1})$
σ	$b_1S_{S,1}/S_Y$
b_D	b_2/b_1
z_D	z/b_1
r_D	r/b_1
t_D	$t\ K_{r,1}/(S_{S,1}b_1^2)$
$s_{D,1}$	$s_1 4\pi b_1\ K_{r,1}/Q$
$s_{D,2}$	$s_2 4\pi b_1\ K_{r,1}/Q$
κ_1	$K_{z,1}/K_{r,1}$
η_1	$\sqrt{(a^2+p)K_{r,1}/K_{z,1}}$
η_2	$\sqrt{(a^2\alpha_{r,2}/\alpha_{r,1}+p)\alpha_{z,2}/\alpha_{z,1}}$
ξ	$\eta_1(b_1S_{S,1}\ K_{z,1})/(S_YK_{r,1}p)$
γ	$\eta_1K_{r,1}/(\eta_2\ K_{r,2})$

$$S_{s,2}\frac{\partial s_2}{\partial t}=\frac{K_{r,2}}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s_2}{\partial r}\right)+K_{z,2}\frac{\partial^2 s_2}{\partial z^2}, \quad (11)$$

where $S_{s,2}$ is the aquitard specific storage, and $K_{r,2}$ and $K_{z,2}$ are its horizontal and vertical hydraulic conductivities. Eq. (11) is solved subject to the following initial and boundary conditions:

$$s_2(r,z,0)=0, \quad (12)$$

$$\lim_{r\to\infty} s_2(r,z,t)=0, \quad (13)$$

$$\lim_{r\to 0} r\frac{\partial s_2}{\partial r}=0, \quad (14)$$

$$K_{z,2}\left.\frac{\partial s_2}{\partial z}\right|_{z=-b_2}=0. \quad (15)$$

Of particular note here is the fact that we have included horizontal flow in the aquitard, which is assumed insignificant in all other leakage theories; a more general aquitard flow problem is considered here. Neglecting horizontal aquitard flow is equivalent to assuming that the horizontal hydraulic diffusivity of the aquitard is negligible compared to its vertical diffusivity. In water-deposited sedimentary formations, the converse is likely to be the case. Justifications for neglecting horizontal aquitard flow come from assuming negligible horizontal hydraulic gradients in the aquitard induced by pumping in the overlying aquifer, or from the steady-state tangent refraction law for the interface between two media, $\tan\theta_1/\tan\ \theta_2=K_1/\ K_2$. Neuman and Witherspoon (1969a) suggest that horizontal head gradients in the aquitard may be negligibly small compared to vertical gradients for cases where the hydraulic conductivity of the aquitard is at least two orders of magnitude smaller than that of the aquifer. However, it should be noted that this was demonstrated only for isotropic aquitards. In anisotropic aquitards with horizontal hydraulic conductivities that are significantly greater than the vertical, horizontal flow may be significant. Our objective is to present a solution that can be used even in cases where the aquitard and aquifer hydraulic conductivities differ by less than two orders of magnitude, and can simulate horizontal flow in the aquitard.

Eqs. (1) and (11) are coupled by imposing the following continuity conditions for drawdown and vertical flux at $z = 0$, their common boundary:

$$s_1(r,0,t) = s_2(r,0,t), \tag{16}$$

$$K_{z,1}\frac{\partial s_1}{\partial z}\bigg|_{z=0} = K_{z,2}\frac{\partial s_2}{\partial z}\bigg|_{z=0}. \tag{17}$$

As in the case of aquifer flow, it is convenient to rewrite the aquitard flow Eq. (11), and the associated initial and boundary conditions, given by Eqs. (12) and (13), in dimensionless form as

$$\frac{\partial s_{D,2}}{\partial t_D} = \frac{\alpha_{D,r}}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial s_{D,2}}{\partial r_D}\right) + \alpha_{D,z}\frac{\partial^2 s_{D,2}}{\partial z_D^2}, \tag{18}$$

$$s_{D,2}(r_D,z_D,0) = 0, \tag{19}$$

$$\lim_{r_D\to\infty} s_{D,2}(r_D,z_D,t_D) = 0, \tag{20}$$

$$\lim_{r_D\to 0} r_D\frac{\partial s_{D,2}}{\partial r_D} = 0, \tag{21}$$

$$\frac{\partial s_{D,2}}{\partial z_D}\bigg|_{z_D=-b_D} = 0, \tag{22}$$

where $s_{D,2} = s_2/H_c$, $\alpha_{D,r} = \alpha_{r,2}/\alpha_{r,1}$, $\alpha_{D,z} = \alpha_{z,2}/\alpha_{r,1}$, $b_D = b_2/b_1$, and $\alpha_{z,i} = K_{z,i}/S_{s,i}$ and $\alpha_{r,i} = K_{r,i}/S_{s,i}$, the vertical and radial diffusivities, respectively, with $i = 1$ for the aquifer and $i = 2$ for the aquitard. Nondimensionalization of the continuity conditions leads to

$$s_{D,1}(r_D,0,t_D) = s_{D,2}(r_D,0,t_D), \tag{23}$$

$$\frac{\partial s_{D,1}}{\partial z_D}\bigg|_{z_D=0} = K_{D,z}\frac{\partial s_{D,2}}{\partial z_D}\bigg|_{z_D=0}, \tag{24}$$

where $K_{D,z} = K_{z,2}/K_{z,1}$, the ratio of the vertical hydraulic conductivity of the aquitard to that of the unconfined aquifer.

Exact solution in Laplace—Hankel transform space

Considering the aquifer flow problem, we apply the Laplace and the zero-order Hankel transforms (see Appendix A for definitions; hereafter, the zero-order Hankel transform will be referred to simply as the Hankel transform) to Eq. (6) and solve the resulting equation subject to the appropriate initial and boundary conditions (see Appendix B for details). The resulting solution can be decomposed as follows (Neuman, 1972)

$$\bar{s}^*_{D,1}(a,z_D,p) = \bar{u}^*_D(a,p) + \bar{v}^*_D(a,z_D,p), \tag{25}$$

where p and a are the Laplace and Hankel transform parameters, respectively, and

$$\bar{u}^*_D(a,p) = \frac{2}{p(p+a^2)}. \tag{26}$$

The analytic inverse Laplace—Hankel transform of $\bar{u}^*_D$ is

$$\mathscr{L}^{-1}\{\mathscr{H}_0^{-1}\{\bar{u}^*_D(a,p)\}\} = u_D(x) = E_1(x), \tag{27}$$

where $E_1(x)$ is the exponential integral (Abramowitz and Stegun, 1972), sometimes called the well function in hydrogeology literature,

$$E_1(x) = \int_x^\infty \frac{e^{-x'}}{x'}\,dx', \tag{28}$$

where x' is a dummy variable and $x = r_D^2/(4t_D)$. Thus, the time-domain solution for the drawdown in the unconfined aquifer is

$$s_{D,1}(r_D,z_D,t_D) = E_1(x) + \mathscr{L}^{-1}\{\mathscr{H}_0^{-1}\{\bar{v}^*_D(a,z_D,p)\}\}, \tag{29}$$

where $\mathscr{L}^{-1}$ and $\mathscr{H}_0^{-1}$ denote the inverse Laplace and Hankel transform operators, respectively (see Appendix A) and $\bar{v}^*_D$ is given by (see Appendix B for details)

$$\begin{aligned}\bar{v}^*_D = &-\frac{\bar{u}^*_D}{\Delta}\{\xi\cosh[\eta_1(1-z_D)] + \sinh[\eta_1(1-z_D)] \\ &+ \gamma\coth(\eta_2 b_D)\cosh(\eta_1 z_D) + \sinh(\eta_1 z_D)\},\end{aligned} \tag{30}$$

where $\gamma = \eta_1/(\eta_2 K_{D,z})$, $\eta_2^2 = (p + \alpha_{D,r}a^2)/\alpha_{D,z}$ and Δ is given by Eq. (B.23) in Appendix B. Additionally, the solution for the double Laplace—Hankel transform of drawdown in the aquitard is given as

$$\begin{aligned}\bar{s}^*_{D,2} = \bar{u}^*_D\Bigg(&\left\{1 - \frac{1}{\Delta}[\xi\cosh(\eta_1) + \sinh(\eta_1) + \gamma\coth(\eta_2 b_D)]\right\}\cosh(\eta_2 z_D) \\ &-\frac{\gamma}{\Delta}[1 - \cosh(\eta_1) - \xi\sinh(\eta_1)]\sinh(\eta_2 z_D)\Bigg).\end{aligned} \tag{31}$$

Considering the solution for flow in the unconfined aquifer and taking the limit as $b_D \to 0$, it can be shown that Eq. (30) reduces to

$$\bar{v}^*_D = -\frac{\bar{u}^*_D\cosh(\eta_1 z_D)}{\xi\sinh(\eta_1) + \cosh(\eta_1)}, \tag{32}$$

and $s_{D,1}$ becomes the solution obtained by Neuman (1972) for unconfined aquifer flow with no leakage. This occurs because as $b_D \to 0$ the no flow condition applied at $z_D = b_D$ (the aquitard base) becomes the boundary condition at $z_D = 0$, the aquifer base. In the limit, as $b_D \to \infty$, $\coth(\eta_2 b_D) = 1.0$ and Eq. (31) reduces to the equation for drawdown response in an unconfined aquifer bounded from below by a semi-infinite aquitard. Hence, the solution developed herein is more general that that of Zlotnik and Zhan (2005) who considered only a semi-infinte aquitard.

Screened observation wells

Eq. (29) is useful for predicting point drawdown response that can be measured with a piezometer. When drawdown response is required for an observation well that is screened over an interval $[z_1, z_2]$ (see Fig. 1), Eq. (29) must be integrated over that interval leading to

$$\langle s_{D,1}(r_D,t_D)\rangle = E_1(x) + \mathscr{L}^{-1}\{\mathscr{H}_0^{-1}\{\langle\bar{v}^*_D(a,p)\rangle\}\}, \tag{33}$$

where $\langle\ldots\rangle$ denotes the averaging operator and $\langle\bar{v}^*_D(a,p)\rangle$ is

$$\langle\bar{v}^*_D(a,p)\rangle = \frac{1}{z_{D,2}-z_{D,1}}\int_{z_{D,1}}^{z_{D,2}} \bar{v}^*_D(a,z_D,p)\,dz_D, \tag{34}$$

where $z_{D,i} = z_i/b_1$ with $i = 1,2$. Carrying out the integration in Eq. (34) leads to

$$\begin{aligned}\langle\bar{v}^*_D(a,p)\rangle = &\frac{\bar{u}^*_D}{\eta_1\Delta}\{\xi\sinh[\eta_1(1-z_D)] + \cosh[\eta_1(1-z_D)] \\ &-\gamma\coth(\eta_2 b_D)\sinh(\eta_1 z_D) - \cosh(\eta_1 z_D)\}\Big|_{z_{D,1}}^{z_{D,2}}.\end{aligned} \tag{35}$$

For the special case of an observation well that is screened across the entire saturated thickness of the unconfined aquifer, i.e. $z_{D,1} = 0$ and $z_{D,2} = 1$, Eq. (35) reduces to

$$\langle \bar{v}^*_D(a,p) \rangle = -\frac{2\bar{u}^*_D}{\eta_1 \Delta}\left[\frac{\xi + \gamma \coth(\eta_2 b_D)}{2}\sinh(\eta_1) + \cosh(\eta_1) - 1\right]. \quad (36)$$

Numerical inversion of double Laplace—Hankel transform solutions

One can, in principle, obtain the analytical inverse Laplace transforms of $\bar{s}^*_{D,1}$ and $\bar{s}^*_{D,2}$ by using the methods of complex contour integration, including the theory of residues (Neuman, 1972). In this work, however, the inverse Laplace and Hankel transforms are obtained numerically. The expressions for $\bar{s}^*_{D,1}$ and $\bar{s}^*_{D,2}$ were evaluated in Laplace—Hankel space using extended precision (greater than Fortran double precision). The inverse Laplace transform was obtained first using the doubly-accelerated Fourier series approach of de Hoog et al. (1982). The inverse Hankel transform (see Appendix A) was approximated by splitting the improper Hankel integral into a finite integral over the interval $0 \leqslant a \leqslant j_{0,n}$, and an infinite integral over the interval $j_{0,n} \leqslant a < \infty$, in a manner similar to that proposed by Wieder (1999). The integral is split at $j_{0,n}$, the nth zero of $J_0\ (ar_D)$. The finite portion of the integral was evaluated using the QXGS automatic integration routine (Favati et al., 1991). Most of the contribution to the total Hankel integral comes from the finite portion of this integral. The infinite integral was approximated by integrating between each $j_{0,n}$ and $j_{0,n+1}$, as n becomes large, using Gauss—Lobatto quadrature for each interval. Finally, partial sums of the areas (which alternate in sign) were accelerated with Wynn's ϵ-algorithm (Antia, 2002; Lee, 1999). Because of the decaying magnitude and alternating signs of the areas between successive zeros, the ϵ-algorithm was very effective.

Theoretical results

Here we present and discuss some key results for the predicted unconfined aquifer and aquitard behavior in a coupled unconfined aquifer—aquitard system, where the aquifer is pumped continuously at a constant rate. In all the figures presented here, $E_1(x)$ is the confined response Theis solution ($x = r_D^2/4t_D$) and $E_1(x_Y)$ is the delayed yield response Theis solution ($x_Y = r_D^2/4t_{D_Y}$, where $t_{D_Y} = K_{z,1}t/b_1S_Y$). The numerically inverted results for both $s_{D,1}$ and $s_{D,2}$ were compared to results from an axisymmetric wedge-shaped MODFLOW simulation (Reilly and Harbaugh, 1993). The finite-difference results agree well with the semi-analytical solution developed here (Fig. 2).

Fig. 3 shows the point piezometer response of the general leaky drawdown response, $s_{D,1}$, for different values of z_D at $r_D = 1.0$ with $\alpha_{D,r} = \alpha_{D,z} = 2.5 \times 10^{-5}$, $b_D = 5.0$, $K_{D,z} = 1.0$, and $S_Y = 0.25$. The solution obtained by Neuman (1972) for the same configuration, but no leakage, is also included. The solution developed herein shows significant departure from that of Neuman (1972) throughout the intermediate time range as well as at late time. However, the

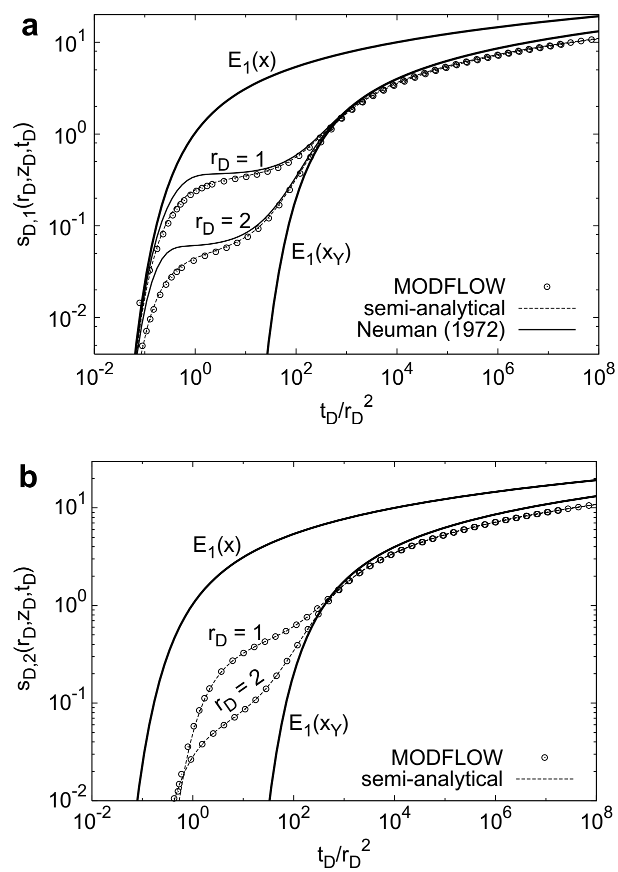

Figure 2 Comparison of MODFLOW and semi-analytic drawdown responses in the aquifer at $z_D = 0.1$ (a) and in the aquitard at $z_D = -0.1$ (b). Both aquifer and aquitard are isotropic.

leakage solution does not reach a steady state as the case would be if the assumptions of classical leakage theory (Hantush and Jacob, 1955) were adopted. In fact the solution developed here follows the curve $E_1(y)$ where $y = (r_D b_1)^2/4\langle\alpha_r\rangle$ and $\langle\alpha_r\rangle$ is the average system radial hydraulic diffusivity. Here, the average radial diffusivity used is $\langle\alpha_r\rangle = b_1\langle K_r\rangle/S_Y$ where $\langle K_r\rangle = (b_1K_{r,1} + b_2K_{r,2})/(b_1 + b_2)$. In Fig. 3 it can be seen that aquifer drawdown generally decreases with increasing dimensionless distance z_D from the aquifer base. However, at very early time drawdown is smaller closer to the base of the unconfined aquifer (the curve for $z_D = 0.0$ crosses over those for $z_D = 0.5$ and $z_D = 0.75$ at early time in Fig. 3). This is attributable to leakage which has the greatest effect at the aquifer base.

Aquitard drawdown response due to pumping in the unconfined aquifer is shown in Fig. 4 for different values of z_D at $r_D = 1.0$. The effect of delayed water table decline is observable in the aquitard very close to its common boundary with the unconfined aquifer. This effect decreases rapidly with increasing depth into the aquitard and, for the example shown here, becomes negligible for $z_D \leqslant -0.25$. Below this level the aquitard behaves as if it were overlain by a confined aquifer. Figs. 5 and 6 are plots of $s_{D,1}$ and $s_{D,2}$, respectively, for different r_D. For the results shown here $K_{z,1}/K_{z,2} = 200$, and the aquifer and the aquitard are isotropic. Fig. 6 is of particular importance as it illustrates that drawdown response in the aquitard has strong variation in

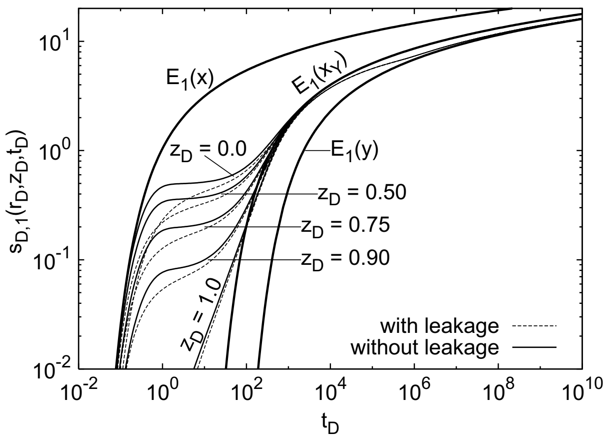

Figure 3 Drawdown response in the unconfined aquifer for different values of z_D ($r_D = 1.0$ and $b_D = 5.0$).

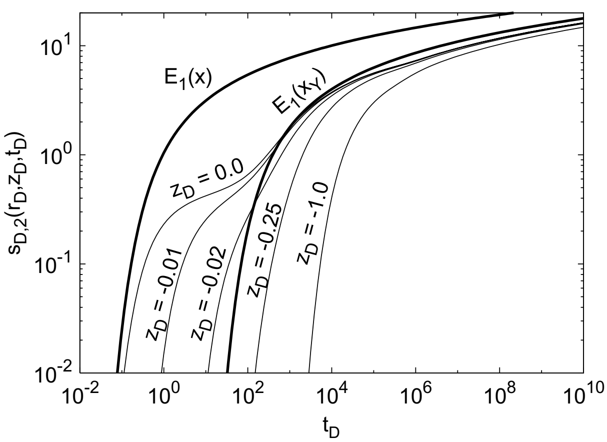

Figure 4 Drawdown response in the aquitard for different values of z_D ($r_D = 1.0$ and $b_D = 5.0$).

the radial direction (a horizontal gradient exists in the aquitard), even for $K_{z,1}/K_{z,2} = 200$. Our solution is very general and can be used to predict aquitard drawdown response under conditions of significant radial flow in the aquitard.

Fig. 7 shows a plot of dimensionless drawdown response, $s_{D,1}$, in an isotropic unconfined aquifer at $(r_D, z_D) = (1.0, 0.1)$ for different values of the vertical hydraulic conductivity ratio, $K_{D,z}$, between the aquifer and an isotropic aquitard. The results were obtained for $b_D = 5.0$ and $S_Y = 0.25$. The results show that as the hydraulic conductivity of the aquitard increases, the effect of leakage increases. For very small values of $K_{D,z}$ (e.g., tight clay, unfractured bedrock), the results shown here conform to those of Ehlig and Halepaska (1976), which showed that the effect of leakage is only noticeable at late time. The solution follows that of Neuman (1972) at early and intermediate time. However, as $K_{D,z}$ increases (e.g. weathered clay, fractured bedrock) the effect of leakage is significant even at early time. An important fact that can be deduced from Fig. 7 is that, if the vertical hydraulic conductivities of the unconfined aquifer and the aquitard differ only by an order of magnitude, leakage can have a significant effect on drawdown response in the aquifer.

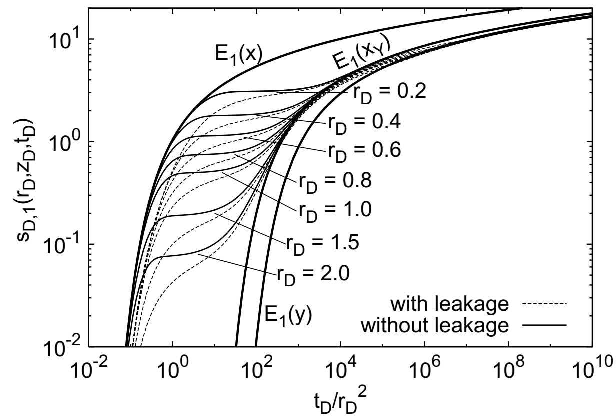

Figure 5 Point drawdown response in the aquifer for different values of r_D ($z_D = 0.1$, $b_D = 5.0$).

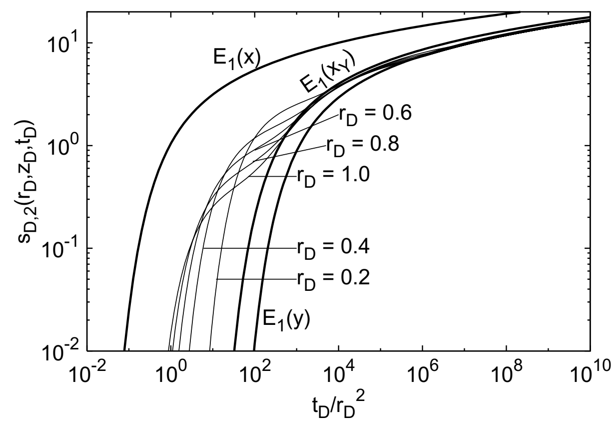

Figure 6 Point drawdown response in the aquitard for different values of r_D ($z_D = -0.1$, $b_D = 5.0$).

The dashed curve in Fig. 7 is the aquifer response for $K_{D,z} = 1.0$ and $S_{s,2} = S_{s,1}$. It can be viewed as the unconfined aquifer response at $z_D = 0.1$ due to a pumping well that partially penetrates (from $z_D = 1.0$ to $z_D = 0.0$) an aquifer

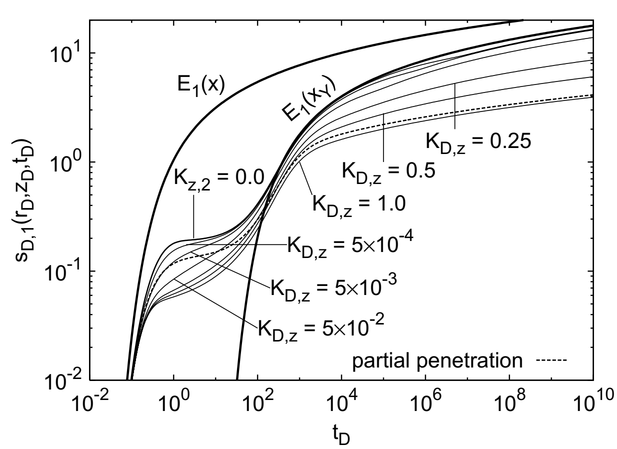

Figure 7 Drawdown response in the unconfined aquifer—aquitard system for different values of $\alpha_{D,r}$ ($\alpha_{D,z} = \alpha_{D,r}$, $r_D = 1.0$, $z_D = 0.1$ and $b_D = 5.0$).

of thickness equal to $b_1 + b_2$. It is the solution that is approached, approximately, at late time as $K_{D,z}$ approaches unity for $S_{s,2} \neq S_{s,1}$; it is included for comparison.

The effect of aquitard anisotropy on drawdown response in an isotropic aquifer with $K_{r,2} = K_{z,2} = 6 \times 10^{-4}$ m/s is shown in Fig. 8. In Fig. 8a the vertical hydraulic conductivity, $K_{z,2}$, of the aquitard is fixed at 6×10^{-6} m/s while the horizontal hydraulic conductivity, $K_{r,2}$, is varied as indicated in the figure. Significant deviation from the solution of Neuman (1972) is observed at both early and late times. However, only late-time response is affected significantly as one changes $K_{r,2}$; early time deviation from the solution of Neuman (1972) is unresponsive to change in $K_{r,2}$. In Fig. 8b the results show the effect of changing $K_{z,2}$ while fixing $K_{r,2}$ at 6×10^{-6} m/s. The results show that changing $K_{z,2}$ while fixing $K_{r,2}$ has significant effects on early time response. The results shown in Fig. 8 suggest that the early time drawdown response in the aquifer is affected by aquitard flow that is predominantly vertical, whereas late-time aquitard flow becomes increasingly horizontal. The results also suggest that for an aquitard with a horizontal hydraulic conductivity significantly larger than the vertical, aquifer response at late time tends to ''near'' steady-state behavior as $K_{r,2}$ approaches $K_{r,1}$.

Fig. 9 shows the drawdown response in the unconfined aquifer for different values of b_D. For very small values of b_D (10^{-3}) leakage is, at early time, manifested as an increase in the apparent specific storage of aquifer and the solution initially follows the solution for $b_D \to \infty$. At intermediate time the solution with leakage practically follows the solution for $b_D = 0$, which is the no leakage case of Neuman (1972). Hence, for such case, leakage is likely to be mis-characterized by over-estimated values of $S_{s,1}$. This may introduce only modest errors. However, as b_D increases, the solution for the leakage case shows increased departure from that of Neuman (1972) at virtually all times and using a large value of $S_{s,1}$ would no longer sufficiently model the departed behavior.

To compare our solution to that of Zlotnik and Zhan (2005) we consider an isotropic unconfined aquifer with $K_{r,1} = K_{z,1} = 10^{-3}$ m/s, $S_{s,1} = 2 \times 10^{-5}$ m^{-1}, $S_Y = 0.2$ and $b_1 = 20$ m, the same hydraulic properties as in Zlotnik and Zhan (2005). In the results shown in Fig. 10a, the unconfined aquifer is underlain by an isotropic semi-infinite aquitard with $S_s = 10^{-3}$ m^{-1} and the effect of aquitard hydraulic conductivity on aquifer drawdown is considered. The results shown are for conductivity values of 10^{-5}, 10^{-6} and 10^{-8} m/s at what Zlotnik and Zhan (2005) referred to as a far field point, $(x, y, z) = (50\text{ m}, 50\text{ m}, 10\text{ m})$. In Fig. 10b the conductivity of the aquitard is fixed at $K_{r,2} = K_{z,2} = 10^{-5}$ m/s and values of $S_{s,2} = 10^{-2}$, 10^{-3}, 10^{-5} and 10^{-7} m^{-1} are considered. The results show that leakage from the aquitard leads to significant deviation from the solution of Neuman (1972) at all times for a relatively highly conductive aquitard $K_{r,2} = K_{z,2} = 10^{-5}$ m/s. The effect diminishes with decreasing aquitard conductivity. The same effect is observed for aquitard specific storage. The results obtained here are comparable to those of Zlotnik and Zhan (2005) at early and intermediate times. However, at late time our conclusion is at variance with that of Zlotnik and Zhan (2005), who computed their solution up to a smaller dimensionless time than is done in this work. It is clear from the figure that leakage leads to significant deviation from Neuman's 1972 solution at late time.

Though the formation below the unconfined aquifer has so far in this work been assumed to be an aquitard, with hydraulic conductivities smaller than those of the aquifer in all directions, it is of interest to note that the solution

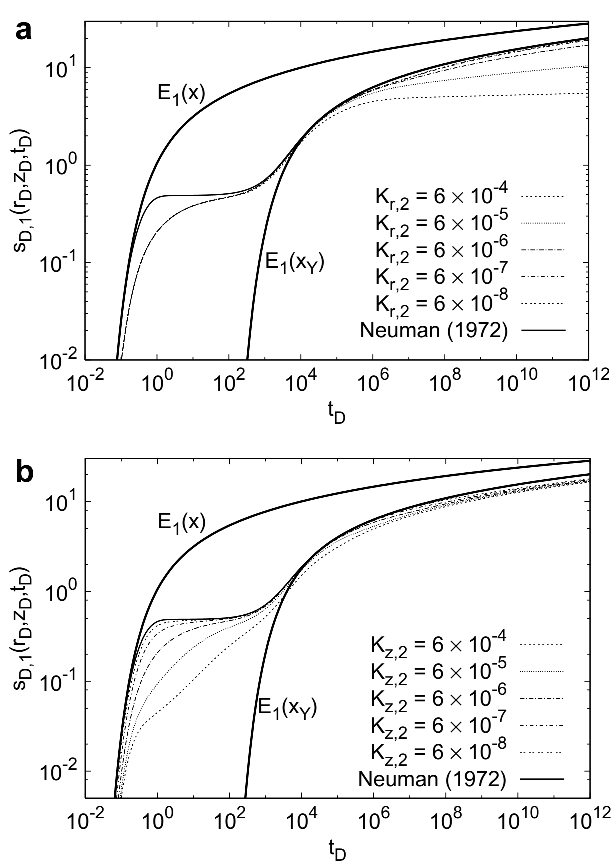

Figure 8 Drawdown response in an isotropic unconfined aquifer ($K_{r,1} = K_{z,1} = 6 \times 10^{-4}$ m/s) underlain with an anisotropic aquitard having (a) $K_{z,2} = 6 \times 10^{-6}$ m/s and (b) $K_{r,2} = 6 \times 10^{-6}$ m/s.

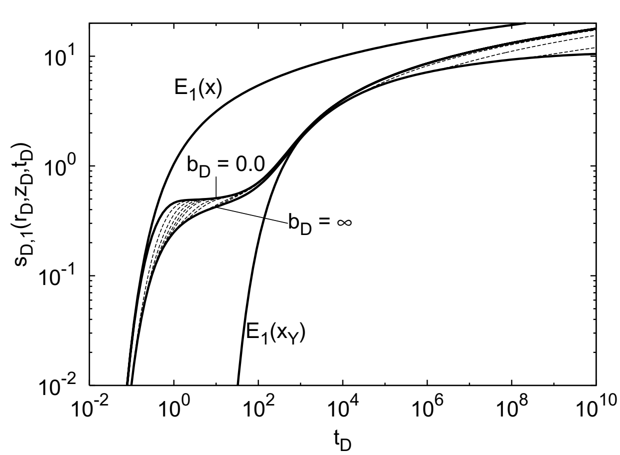

Figure 9 Drawdown response in aquifer at ($z_D = 0.1$, $r_D = 1.0$) for different values of b_D.

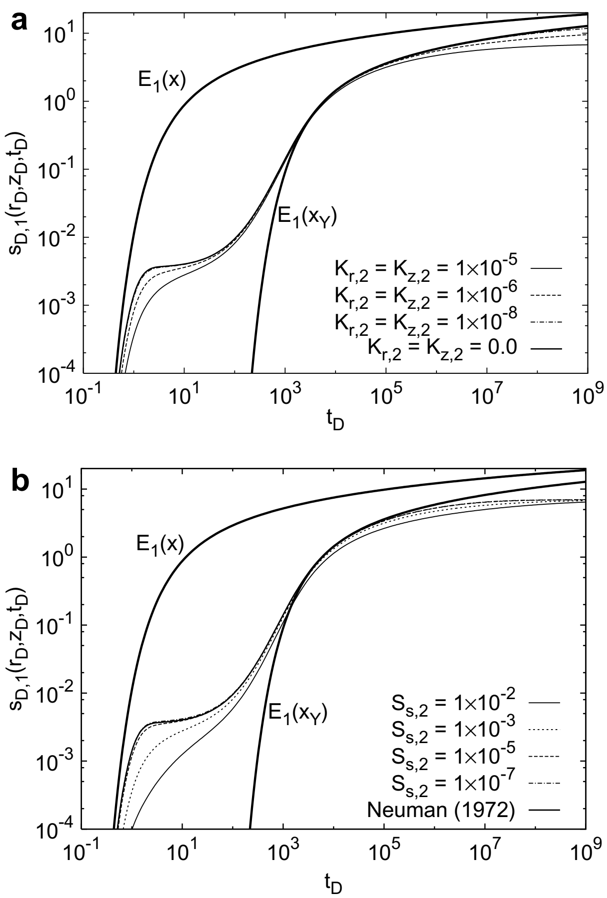

Figure 10 Drawdown response in aquifer at (x_D, y_D, z_D) = (0.25, 0.25, 0.50) for different values of $K_{r,2} = K_{z,2}$ (top), and $S_{s,2}$ (bottom).

developed herein is not limited to this scenario. Our solution allows for more conductive formation below the unconfined aquifer because we do not neglect horizontal flow in the formation below the aquifer. To illustrate this, we consider an isotropic unconfined aquifer with $S_{s,1} = 10^{-5}\,\mathrm{m}^{-1}$ and $S_Y = 0.2$, underlain with a formation having $S_{s,2} = 10^{-3}\,\mathrm{m}^{-1}$. For the case of the lower formation being less conductive than the aquifer, we set aquifer conductivity at $K_{r,1} = K_{z,1} = 10^{-3}$ m/s and that of the aquitard at $K_{r,2} = K_{z,2} = 10^{-5}$ m/s, whereas for a more conductive lower formation, $K_{r,1} = K_{z,1} = 10^{-5}$ m/s and $K_{r,2} = K_{z,2} = 10^{-3}$ m/s; the results are shown in Fig. 11. When the underlying aquifer is more conductive the effects of leakage are much more significant, a scenario which could not be tested using previous unconfined-leaky models. Comparing Fig. 11 with the others, it is clear that a steady state is attained more quickly when the system consists of two aquifers than when the system is an aquifer underlain by an aquitard.

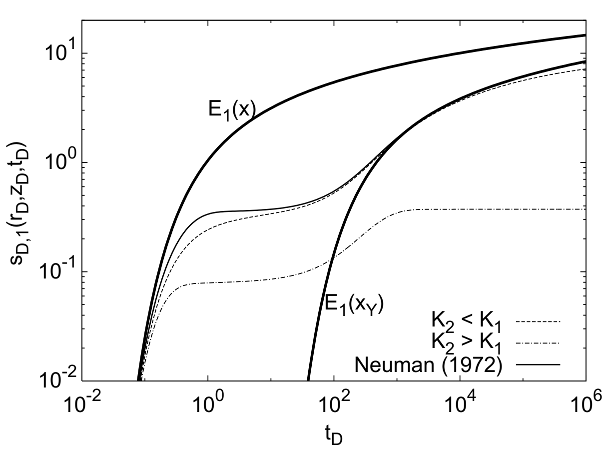

Figure 11 Comparison of drawdown response in aquifer at (r_D, z_D) = (1.0, 0.5) for $K_2 > K_1$ with the case of $K_2 < K_1$.

Summary

Leakage can be as important in unconfined aquifers as it is in confined aquifers. Results presented here constitute an attempt to solve the leakage problem for unconfined aquifers analytically. We have presented semi-analytical solutions, the exact analytical solutions in the double Laplace—Hankel transform space are inverted numerically. Although analytical expressions are not given for the inverse transforms of the double Laplace—Hankel transforms of the unconfined aquifer and aquitard drawdown, widespread availability of nonlinear least squares routines and fast computers provide for quick computation of system behavior as well as for estimation of system parameters using the exact solutions for the double Laplace—Hankel transforms of drawdown developed here.

In the solution obtained by solving the unconfined aquifer and aquitard flow problems simultaneously, horizontal flow in the aquitard was not neglected and unconfined aquifer flow was treated in the manner of Neuman (1972). We used drawdown and flux continuity conditions at the contact between the aquifer and the aquitard, removing the need to make the assumptions adopted in the classical and modified leakage theories of Hantush and Jacob (1955) and Hantush (1960). An important result obtained here is that leakage can lead to significant departure from Neuman's 1972 solution even at early time, which is at variance with the results of Ehlig and Halepaska (1976) that showed departure only at late time. The solution developed in this work can also be used to estimate hydraulic parameters of aquitards overlain by unconfined aquifers. It is more general than that developed by Zlotnik and Zhan (2005) as it allows for horizontal flow in the formation below the unconfined aquifer. It also allows for an underlying formation of both finite and semi-infinite vertical extent. We have not developed type-curve procedures for estimation of aquifer and aquitard parameters due to the large number of parameters involved and the increased availability of automatic parameter estimation routines.

Appendix A

The zero-order Hankel transform, $f_0^*(a)$, of a function, $f(r_D)$, is given by

$$\mathscr{H}_0\{f(r_D)\} = f_0^*(a) = \int_0^\infty r_D J_0(a r_D) f(r_D)\,\mathrm{d}r_D, \tag{A.1}$$

where a is the real-valued Hankel parameter and J_0 is the zero-order Bessel function of the first kind. The inverse Hankel transform of $f_0^*(a)$ is defined as

$$\mathscr{H}_0^{-1}\{f_0^*(a)\} = f(r_D) = \int_0^\infty aJ_0(ar_D)f_0^*(a)\,da. \tag{A.2}$$

A particular relation, adopted from Neuman and Witherspoon (1968), used in this work, is

$$\mathscr{H}_0\left\{\frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial \bar{s}_D}{\partial r_D}\right)\right\} = -a^2\bar{s}_D^* - \lim_{r_D\to 0} r_D\frac{\partial \bar{s}_D}{\partial r_D}. \tag{A.3}$$

Appendix B

To obtain the general solution to the dimensionless flow problem for the unconfined aquifer, we follow the work of Neuman (1972) and decompose $s_{D,1}$ as

$$s_{D,1}(r_D, z_D, t_D) = u_D(r_D, t_D) + v_D(r_D, z_D, t_D), \tag{B.1}$$

where u_D satisfies

$$\frac{\partial u_D}{\partial t_D} = \frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial u_D}{\partial r_D}\right), \tag{B.2}$$

subject to

$$u_D(r_D, 0) = 0, \tag{B.3}$$

$$\lim_{r_D\to\infty} u_D(r_D, t_D) = 0, \tag{B.4}$$

$$\lim_{r_D\to 0} r_D\frac{\partial u_D}{\partial r_D} = -2. \tag{B.5}$$

The component v_D satisfies the equation

$$\frac{\partial v_D}{\partial t_D} = \frac{1}{r_D}\frac{\partial}{\partial r_D}\left(r_D\frac{\partial v_D}{\partial r_D}\right) + \kappa_1\frac{\partial^2 v_D}{\partial z_D^2}, \tag{B.6}$$

subject to

$$v_D(r_D, z_D, 0) = 0, \tag{B.7}$$

$$\lim_{r_D\to\infty} v_D(r_D, z_D, t_D) = 0, \tag{B.8}$$

$$\lim_{r_D\to 0} r_D\frac{\partial v_D}{\partial r_D} = 0, \tag{B.9}$$

$$-\left.\frac{\partial v_D}{\partial z_D}\right|_{z_D=1} = \frac{1}{\alpha_{D,Y}}\left(\frac{\partial u_D}{\partial t_D} + \left.\frac{\partial v_D}{\partial t_D}\right|_{z_D=1}\right). \tag{B.10}$$

Taking the Laplace and Hankel transforms of Eq. (B.2) and solving subject to the initial and boundary conditions given in Eqs. (B.3)—(B.5) leads to the Theis solution (26). Taking the Laplace and Hankel transforms of Eq. (B.6) subject to initial and boundary conditions given in Eqs. (B.7) and (B.8) leads to

$$\frac{\partial^2 \bar{v}_D^*}{\partial z_D^2} - \eta_1^2\bar{v}_D^* = 0, \tag{B.11}$$

where $\eta_1^2 = (p + a^2)/\kappa_1$. Eq. (B.11) has a general solution of

$$\bar{v}_D^*(\eta_1, z_D) = c_1 e^{\eta_1 z_D} + c_2 e^{-\eta_1 z_D}. \tag{B.12}$$

Applying the boundary condition (at the water table) given by Eq. (B.10), one obtains the following linear equation for the constants (in z_D) c_1 and c_2:

$$-(\xi + 1)e^{\eta_1}c_1 + (\xi - 1)e^{-\eta_1}c_2 = \bar{u}_D^*, \tag{B.13}$$

where $\xi = \eta_1\alpha_{D,Y}/p$.

To obtain the general solution for aquitard drawdown response, we take the Laplace and Hankel transforms of Eq. (18) subject to the initial and boundary conditions given by Eqs. (19)—(21), leading to

$$\frac{\partial^2 \bar{s}_{D,2}^*}{\partial z_D^2} - \eta_2^2\bar{s}_{D,2}^* = 0, \tag{B.14}$$

where $\eta_2^2 = (p + \alpha_{D,r}a^2)/\alpha_{D,z}$. The general solution for Eq. (B.14) is

$$\bar{s}_{D,2}^*(\eta_2, z_D) = c_3 e^{\eta_2 z_D} + c_4 e^{-\eta_2 z_D}. \tag{B.15}$$

Applying the no flow boundary condition at the base of the aquitard leads to the following relationship for the constants c_3 and c_4,

$$c_3 e^{-\eta_2 b_D} - c_4 e^{\eta_2 b_D} = 0. \tag{B.16}$$

Applying the head and flux continuity conditions at $z_D = 0$ given by Eqs. (23) and (24), yields the final linear equations required to determine the constants c_1 through c_4,

$$-c_1 - c_2 + c_3 + c_4 = \bar{u}_D^*, \tag{B.17}$$

$$-\gamma c_1 + \gamma c_2 + c_3 - c_4 = 0, \tag{B.18}$$

where $\gamma = \eta_1/(\eta_2 K_{D,z})$. Solving the system of equation given by Eqs. B.13,B.16,B.17 and B.18 yields

$$c_1 = -\frac{\bar{u}_D^*}{2\Delta}[(\xi - 1)e^{-\eta_1} + \gamma\coth(\eta_2 b_D) + 1], \tag{B.19}$$

$$c_2 = -\frac{\bar{u}_D^*}{2\Delta}[(\xi + 1)e^{\eta_1} + \gamma\coth(\eta_2 b_D) - 1], \tag{B.20}$$

$$c_3 = \frac{\bar{u}_D^*}{2}\{1 - [(\xi - \gamma)\cosh(\eta_1) + (1 - \xi\gamma)\sinh(\eta_1) + \gamma(\coth(\eta_2 b_D) + 1)]/\Delta\}, \tag{B.21}$$

$$c_4 = \frac{\bar{u}_D^*}{2}\{1 - [(\xi + \gamma)\cosh(\eta_1) + (1 + \xi\gamma)\sinh(\eta_1) + \gamma(\coth(\eta_2 b_D) - 1)]/\Delta\}, \tag{B.22}$$

where

$$\Delta = [\xi\gamma\coth(\eta_2 b_D) + 1]\sinh(\eta_1) + [\gamma\coth(\eta_2 b_D) + \xi]\cosh(\eta_1). \tag{B.23}$$

The final solution in Laplace—Hankel transform space is found by substituting the expressions for the constants, (B.19), (B.20), (B.21), (B.22), into the components of the aquifer, (B.12) and (26), and aquitard drawdown (B.15).

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