IN SITU DETERMINATION OF SOIL STIFFNESS AND DAMPING

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ABSTRACT: Determination of in situ dynamic soil properties is fundamental to the prediction of the seismic behavior of foundations and soil embankment structures. Both elastic (stiffness) and inelastic (damping) values are required for computational analysis. To be of value to engineers, the geophysical inversion should employ the same soil model as used in the dynamic analysis software. Current engineering practice employs a Kelvin-Voigt model (spring in parallel with dashpot). The relevant wave equation is a third-order partial differential equation. This paper demonstrates how to collect in situ field data and solve for stiffness (scaled shear) and damping values by a method consistent with this constitutive model. Measurements of seismic wave amplitude decay and velocity dispersion are simultaneously inverted for the required stiffness and damping values. These in situ stiffness and damping values are directly comparable to those obtained by resonant column measurements in the laboratory. Furthermore, the results may be directly input into currently available engineering software to provide values of stiffness and viscous damping. This paper includes both synthetic (finite difference) and field data examples that illustrate the method.

INTRODUCTION

It is important to identify the steps common to all in situ determinations of dynamic soil properties. These steps are:

1. Recording of in situ wave propagation data.
2. Measurement of recorded waveform data.
3. Calculation of dynamic soil properties from the measurements.
4. Mapping errors in observed measurements to errors in calculated soil properties.

The first step is accomplished by field experiments such as down-hole, cross-hole, or surface seismic recording. In the second step, in which measurements are taken from recorded field data, the measured quantities might be, for example, travel times and distances in elastic analysis. Note that these measurements are not the dynamic properties of interest. Dynamic properties are calculated, not measured. Calculation of dynamic soil properties, the third step, is done under a constitutive model and corresponding governing differential equation. In fact, the concept of each dynamic property is intimately bound to the assumed constitutive model. There is no concept of a soil damping value in an elastic constitutive model. When SH-wave velocity is accounted for in an elastic model, there is only one soil property, the stiffness of the soil. If damping is actually present, some portion of the velocity will incorrectly be attributed to stiffness. Thus, the computation of a shear modulus from wave velocity will be in error if significant viscous damping is present.

It is interesting to note that the same set of recorded waveform data may be subjected to different measurement and calculation procedures, depending on the model assumed. Changing either the constitutive model or the measurements taken from the data leads to different determinations of soil properties for the same field experiment. The different types of field experiments only fix the relevant boundary conditions and wavefield sampling.

The specific calculations in steps 3 and 4 are done using a mathematical formalism known as inverse theory. Inverse theory is well documented. A good text on the subject is that written by Menke (1989). Further examples of inverse theory may be found in the work of Lines and Levin (1988).

REVIEW OF IN SITU METHODS

Various researchers have conducted in situ determination of rock and soil dynamic properties, using a variety of experimental protocols, over the last three decades. An example of the least invasive method is a surface wave experiment conducted with both source and receivers on the surface of the earth. Rayleigh waves (a combination of both compressional and shear waves) have proved particularly useful in this regard (Nazarian and Stokoe 1984). The inverse method, spectral analysis of surface waves, is typically used to determine the shear-wave velocity in a horizontally layered earth under an elastic constitutive model. Measurements taken from the surface waves yield velocity dispersion as a function of frequency. The dispersion curves are then inverted to determine actual soil velocities. In an elastic model, this dispersion is entirely due to the soil layering (configurational dispersion). It would be very challenging to adapt the methods contained in this paper to surface waves. In use of the Kelvin-Voigt model, the challenge would be to separate the observed dispersion into two components, configurational and inelastic. Some attempts at surface measurement of attenuation have been tried using a different wave type. Stoll (1983) documents the use of refracted acoustic waves in a marine environment to solve for a complex modulus.

In sandy or fine grained soils, the seismic cone penetration test (SCPT) can be used without actually drilling a borehole (Robertson et al. 1985). In this test, a standard cone penetrometer is modified to include a horizontally oriented geophone. The modified penetrometer is driven into the soil in the usual fashion for CPT. Periodically, the penetrometer is halted at different depths and a surface source is excited. Thus, shear waves are recorded along a vertical propagation path. Usually, this path is perpendicular to any horizontally layered boundaries. Although the majority of SCPT calculated soil properties have involved elastic assumptions, there should be no impediment to using SCPT data with the inversion method presented in this paper.

One practical limitation of the SCPT method is that soils with gravel or cobbles can refuse the penetrometer. Therefore, a down-hole method is usually preferred when these coarse grained soils are present. An example of the down-hole method in gravelly soils may be found in the work of Kokusho et al. (1995). Kokusho found that SH-wave velocity did not necessarily depend on void ratio according to traditional expectations for sandy soils. Rather, the found that SH-wave velocity is highly dependent on gradation. Readers should keep

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in mind the finding that gravelly soils are anomalous when reading the final field example contained here.

The most expensive method used to determine in situ dynamic soil properties is the cross-borehole experiment. The expense is a result of requiring three or more boreholes to be drilled. Typically a source is lowered in one borehole and receivers record horizontally propagating shear-waves in the other two boreholes. The reader is referred to standard ASTM D-4428 for a description of the typical method and elastic velocity analysis. One common application of the crosshole method has been to evaluate in situ dynamic compaction programs (Dise et al. 1994).

The cross-hole method was recently used to great advantage by Salgado et al. (1997). They demonstrated that large strain measurements can be made with this configuration (if one of the receiver boreholes is close enough to the source hole). This is an extremely useful result, since it is the first in situ method to actually determine the strain amplitude dependence of the shear-modulus.

One might hope to apply the inversion method of this paper to cross-borehole data. However, since soil layering is typically horizontal, any configurational dispersion (wave guide effect) might be difficult to separate from inelastic dispersion due to viscous effects. This does not mean that down-hole methods are free from any complications. It is true that scattering effects across layer boundaries interfere with the down-hole measurement of attenuation. However, in the down-hole configuration, more sampling within a layer is often possible. This permits better statistical averaging within a layer than is generally possible with only two receiver positions in the cross-hole method. The choice between cross-hole and down-hole work is often a choice between dealing with complications in measuring dispersion and complications in measuring attenuation, respectively. Down-hole work is generally less expensive.

**DOWN-HOLE EXPERIMENT**

The writer's field data were acquired using a moveable down-hole three-component clamped geophone and a stationary three-component reference phone. The reference phone is used to compensate for small variations in source strength, spectral content, and any possible trigger variations. Fig. 1 shows the writer's typical arrangement of a down-hole experiment. The borehole and reference phone are usually about one meter from the source. This provides a nearly vertical propagation path for the waves in all but the shallowest levels. The reference phone is buried about 10 cm to protect it from noise sources at the surface. Multiple source efforts are acquired and summed at each borehole station. Typically, five efforts significantly overcome random background noise. Borehole geophone stations are acquired at regular intervals, 0.5 or 0.25 m, depending on the detail desired.

Fig. 2 shows the design of two SH-wave sources built by the writer. In Fig. 2(a), the source is constructed from a 1 m length of railroad tie. Sledge hammers are pivoted by angle iron supports and activated by ropes. Only one hammer is used at a time, while the other hammer is tied off from the beam. The wood beam is struck directly by the broadside edge of the hammer. The soft hammer blows are highly repeatable and produce a useful bandwidth from about 15 to 150 Hz. Sand bags may be placed on the railroad tie to provide a static load which couples the beam to the soil. Coupling is further enhanced by angle iron cleats mounted on the base of the beam. The angle iron pivots and rope activation system protect the operator from back injury that might result from free swings of the hammers. Each geophone level is acquired with two source polarizations so that SH-waves may be confirmed and enhanced by a subtraction process.

The other design, shown in Fig. 2(b), is lighter weight and requires no static hold down load other than the weight of the mechanism itself (about 45 lbs). It is nailed to the soil and employs the same soil key lock on the base of the timber. Further, the blows are delivered at 45° from the horizontal, producing a dynamic hold down force. The single hammer pivot point can be rotated to either side, eliminating the need for two hammers. Horizontal data acquired with both sources are quite similar, but the source shown in Fig. 2(b) has the added advantage of an extra amount of vertical motion, which can be used in P-wave studies.

The data of interest are acquired on the horizontal component phones. A Bison 9048 series engineering seismograph was used to collect the data. Because the tool may rotate and change orientation as it moves up the hole, the two horizontal borehole signals must be rotated to a standard phone orientation following hodogram analysis of the particle motion. A hodogram is a two-dimensional plot of the ground motion time history in the horizontal plane. Based on the measured azimuth of the linear particle motion, a coordinate rotation is performed to project the entire particle motion onto a single horizontal channel. This is the mathematical equivalent of rotating the tool so that one of the phones is aligned parallel to the source polarization.

**CHOOSING A CONSTITUTIVE MODEL**

Over the years, the calculation of the actual dynamic soil properties has been done under differing constitutive models (Elastic, Kelvin-Voigt, and Maxwell being the most common). Kudo and Shima (1970) provide a concise review of some of the early efforts. As they point out, the lack of a definitive consensus on the appropriate constitutive model has been due...
in large part to the limited bandwidth of the observations. Under conditions of limited band width, almost any model produces acceptable results.

On the other hand, engineering practice has tended toward the use of the Kelvin-Voigt model for consolidation and soil dynamics. Computational examples include SHAKE (Schnabel et al. 1972) and DESRA2 (Lee and Finn 1982). Furthermore, laboratory measurements by resonant column techniques also invoke the Kelvin-Voigt soil model (Hardin 1965). The laboratory measurements may be done at comparable frequencies, strain magnitudes, and stress conditions to those in a down-hole experiment. Further, determination of soil properties is often done using the same Kelvin-Voigt model. Therefore, it is easy to compare laboratory results with the in situ measurements described in this paper. This is true regardless of the type of boundary conditions invoked for the passive end of the oscillator (fixed or free). The reader is referred to the work of Drnevich et al. (1978) and to standard ASTM D-4015-92.

The major motivation for this work was to refine the acquisition and processing of down-hole data so that the results would be of direct value to those civil engineers using software employing the Kelvin-Voigt model. Since this model predicts both attenuation and body wave dispersion, the method evolved into the use of both of these measures for a simultaneous inversion leading to wave equation coefficients. Therefore, those engineers who use this software will find that the method provides exactly those values needed for inclusion in the stiffness and damping matrices.

**KELVIN-VOIGT MODEL**

The equation of motion is derived from the spring-mass-dashpot analog model shown in Fig. 3. Consider just one of the spring-mass-dashpot elements in this discrete model of a continuum. Such a single element is a single-degree-of-freedom model, which could be used to represent the fundamental mode of excitation in a resonant column experiment. The spring provides the stiffness (force proportional to displacement), and the dashpot represents the viscous damping (force proportional to particle velocity). Measurement of amplitude decay or bandwidth in a resonant column oscillator under this model yields stiffness and damping values, which may be directly related to those determined in this paper.

When considering the multiple-degree-of-freedom chain of spring-mass-dashpot elements (assuming for the moment all elements are identical), the finite difference equation of motion is found by summing the stiffness and damping forces. Thus, for the jth element in the chain,

$$\frac{\partial^2 u_j}{\partial t^2} = \frac{\Delta x^2}{m} \frac{\Delta^2 u_j}{\Delta x^2} + \frac{\Delta x^2}{m} \frac{\Delta^2 u_j}{\Delta x^2}$$

(1)

where $u_j$ and $v_j$ are the particle displacement and particle velocity for the jth mass, measured in meters. The spring constant is $k$, and the dashpot damping is $d$. The element spacing is $\Delta x$. This equation is often cast in matrix form, as was done in generating synthetic data to test the inverse method presented in this paper.

In the limit of a continuum, difference equation (1) becomes

$$\frac{\partial^2 u}{\partial t^2} = C_1 \frac{\partial^2 u}{\partial x^2} + C_2 \frac{\partial^3 u}{\partial x^3}$$

(2)

where $u$ is particle displacement, $x$ is the spatial coordinate, and $t$ is time. Constant $C_1$ is the stiffness coefficient (spring), and $C_2$ is the viscous damping coefficient (dashpot). The ratio of $C_2$ (m/s²) to $C_1$ (m/s²/s) is the relaxation time in seconds.

Biot (1941), referring to Terzaghi’s earlier work, recalled a useful analogy for the Kelvin-Voigt solid. The relaxation time is a measure of how long a permeable sponge, squeezed underwater, takes to resume its initial equilibrium. This depends on the combination of the stiffness of the fibers (spring) and the permeability permitting the fluid to return to the previously squeezed pores (viscous damping).

For a continuum, the coefficients $C_1$ and $C_2$ are given in terms of shear modulus ($G$), mass density ($\rho$), and absolute viscosity ($\eta$) of the soil model. Therefore, the mapping between the difference equation coefficients and those of the governing differential equation are

$$\frac{\Delta x^2}{m} \rightarrow \frac{G}{\rho} = C_1$$

(3)

$$\frac{\Delta x^2}{m} \rightarrow \frac{\eta}{\rho} = C_2$$

(4)

Modeling soils with greater detail is possible. For example, Biot (1962) and Gajo (1995) characterized the soil as a two-phase medium, keeping fluid and frame motions distinct. Gajo computed the theoretical effects of damping on transient waveforms. However, separate recording of frame and fluid motion is beyond this writer’s capability, and the introduction of greater detail results in an increase in the number of unknowns, which often leads to non-uniqueness in the solution to the inverse problem. For these reasons, this writer has chosen to reduce the problem to only two unknowns, stiffness and damping, which admittedly are functionally dependent on other parameters such as fluid viscosity, frame porosity, permeability, and the densities of the fluid and grains.

Eq. (2) reduces to the elastic wave equation in the absence of damping ($C_2 = 0$),

$$\frac{\partial^2 u}{\partial t^2} = C_1 \frac{\partial^2 u}{\partial x^2}$$

(5)

Ideally, both $C_1$ and $C_2$ are constants independent of frequency. Furthermore, both $C_1$ and $C_2$ should be properties of the medium. In reality, other factors also are relevant to the establishment of effective values for these two "constants." For example, in the elastic limit, $C_1$ is the square of the phase velocity of the wave. The phase velocity depends on the shear modulus, which in turn has been shown empirically to depend on a number of parameters which include the magnitude of the strain, the stress field, and the void ratio of the material (Hardin and Richart 1963).

It is difficult to underestimate the appeal of the Kelvin-Voigt (viscous damping) model. While it is clear that water saturated soils should exhibit a viscous interaction with the frame, it is far less obvious that a dry soil should exhibit viscous behavior. (Grain to grain contact friction is the more likely dominant cause of energy dissipation.) Therefore, Hardin (1965) found
it necessary to vary viscosity as a function of frequency for dry specimens. In short, he suggested that the product of viscosity with frequency divided by shear modulus should be held constant. Hardin’s suggestion has the effect of eliminating velocity dispersion, linears the variation of decay, α, with frequency, and establishes a constant Q (quality factor) medium. The writer has decided not to vary C1 in a similar fashion, because doing so would suggest that a different constitutive relationship might be more appropriate. One alternative for dry soils might be a micromechanical approach invoking a contact law (Cascante and Santamarina 1996). Attempts to apply the writer’s procedure within the vadose zone have not supported the Kelvin-Voigt model. (Holding C1 constant in dry soils does not seem appropriate.) For that reason, the only field examples presented here are taken from below the water table.

Application to Soil Profile

The practical matter of dividing the subsurface into layers involves three considerations:

1. Each layer thickness must be great enough for dispersion and attenuation to be measured with some confidence. Multiple sample points within a layer are needed to determine error bars.
2. Each layer should not be so thick that a significant variation of stress occurs across the layer. This problem is greatest in the first few meters below the surface.
3. The layer boundaries should avoid combining significantly different soil types in the same layer, subject to the resolution limits imposed by the available source band width (see next section).

In the writer’s experience a nominal layer thickness of about 5 m with 0.5 to 0.25 intervals between geophone stations seems to satisfy the above requirements for layers in the 5 to 25 m depth range.

Limits on Resolution

The ultimate limits on spatial resolution depend on the bandwidth and frequencies present in the source radiation. This is true for all seismic methods—down-hole, cross-hole, or conducted on the surface. In the context of typical soil velocities, and the writer’s hammer sources, the available wavelengths will range from 50 m to 1 m. Although increasing the high-frequency content might appear desirable in the context of resolution, it must be cautioned that the relaxation mechanism will change as the pore fluids begin to move with the frame. This change begins at about 100 Hz for many soils.

The third principle in the preceding section suggests that the relevant elementary volume should avoid inclusion of different soil types. This is only true if the distinct soil types occur in layers which are thick in the context of the dominant wavelengths radiated by the source. Even relatively extreme variations in soil types may be combined into a single relevant elementary volume, if they are thin as measured by the available seismic wavelengths (i.e., below the resolution of the source radiation).

FORWARD PROBLEM

For any given values of C1 and C2, one must be able to calculate the frequency dependent attenuation and body wave dispersion. The needed formulas may be derived by substituting the trial solution,

\[ u(x, t) = \exp(-\alpha x) \cdot \cos(\beta x - \omega t) \] (6)

into (2). The complex part of wavenumber, \( \alpha \), is an attenuation coefficient (measured in 1/meters), a function of frequency. The real part of wavenumber is \( \beta \), and \( \omega \) is frequency. The result is

\[
(2C_1\alpha \beta - C_2(-\alpha^2 \omega + \omega^2\beta))\sin(\beta x - \omega t) + [C_1(\alpha^2 - \beta^2) - 2C_2\alpha \omega \beta + \omega^2\beta]\cos(\beta x - \omega t) = 0
\] (7)

Because (7) must be true for all time values, \( t \), and space values, \( x \), the coefficients of the sine and cosine terms must equal zero. This results in two equations in two unknowns:

\[
\begin{bmatrix}
(2\alpha \beta) & (-\alpha^2 \omega + \omega^2 \beta) \\
(\alpha^2 - \beta^2) & -(2\alpha \omega \beta)
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
-\omega^2
\end{bmatrix}
\] (8)

Solving (8) for \( C_1 \) and \( C_2 \) produces

\[
C_2 = \frac{2\alpha \omega \beta}{(\beta^2 + \omega^2)^2}
\] (9)

and

\[
C_1 = \frac{(\beta^2 - \alpha^2) \omega^2}{(\beta^2 + \alpha^2)^2}
\] (10)

Solving (9) and (10) for attenuation (\( \alpha \)) and phase velocity (\( c = \omega / \beta \)) leads to the forward equations needed in the design of a least squares inversion algorithm, in which \( \alpha \) and \( c \) are measured from the down-hole seismic waveforms as a function of frequency.

Let’s begin by taking the ratio of (9) to (10). Solving the resulting quadratic formula for wavenumber, \( \beta \), yields

\[
\beta = \frac{\alpha}{2C_2 \omega} \cdot D
\] (11)

where

\[
D = 2[C_1 + \sqrt{C_1^2 + \omega^2 C_2^2}]
\] (12)

Substituting (11) for \( \beta \) in (10) and solving for the attenuation, \( \alpha \), produces

\[
\alpha = \frac{4\sqrt{D_0} \omega^3 C_2}{(2 \omega c^2 + D^2)}
\] (13)

Note that if the damping coefficient, \( C_2 \), vanishes, there is no attenuation. Furthermore, the constant, nonzero value of \( C_2 \) predicts the frequency dependent behavior of attenuation, \( \alpha \).

Replacing wavenumber, \( \beta \), with \( \omega / c \) and solving for phase velocity, \( c \), one obtains

\[
c = \frac{2\omega^3 C_2}{D \alpha}
\] (14)

The frequency dependent behavior of phase velocity depends on both \( C_1 \) and \( C_2 \). If damping, \( C_2 \), vanishes, the phase velocity reduces to the elastic case and becomes a constant (no dispersion). Applying L’Hospital’s rule (since \( \alpha \) also vanishes), one obtains a constant phase velocity for the zero damping case:

\[
c = \sqrt{C_1}
\] (15)

Eqs. (13) and (14) provide the forward equations needed for the inversion algorithm. The section on inversion shows how measurements of both \( \alpha \) and \( c \), made at selected frequencies, are then jointly inverted for the constants \( C_1 \) and \( C_2 \), stiffness and damping. Before consideration of the inverse problem, however, the mathematical relationships connecting \( C_2 \) to other common expressions of damping will be described.
RELATIONSHIP OF C, TO OTHER FORMS

While the Kelvin-Voigt model appears frequently in the literature, a number of forms have been chosen to express viscous damping under this model. These forms include complex modulus, loss tangent, loss angle, and damping ratio. For the benefit of those readers who work with these other forms, the following summary is given.

One may define a complex shear modulus of the form

\[ G^* = G_r + iG_i \]  

with real part, \( G_r \), and complex part, \( G_i \). This modulus is related to the wave equation coefficients by

\[ C_1 = \frac{G_r}{\rho}, \quad C_2 = \frac{G_i}{\omega\rho} \]  

The complex shear modulus, \( G^* \), varies as a function of frequency in the Kelvin-Voigt model. This variation is linear and entirely due to the complex part, \( G_r \), and its relation to \( C_1 \) in (17).

An alternative form may be computed from the complex shear modulus. This expression is the loss tangent. The loss tangent is given by

\[ \tan(\delta) = \frac{G_i}{G_r} \]  

where \( \delta \) is the loss angle. It follows from (17) that

\[ \tan(\delta) = \omega \left( \frac{C_2}{C_1} \right) = \omega T_r \]  

Here \( T_r \) is the relaxation time. Loss tangent (or loss angle) will also vary with frequency. In the Kelvin-Voigt model, the variation is linear with the slope being the relaxation time.

Finally, resonant column workers often employ the concept of damping ratio (Drnevich 1978). Damping ratio \( D_r \) (the ratio between any value of damping to critical damping) is given by

\[ D_r = \frac{C_1\omega \rho}{2G_r} \]  

where \( \omega_0 \) is the resonant frequency. The resonant frequency is given by the root of the ratio of the equivalent spring constant to polar moment of inertia in a resonant column experiment (Drnevich et al. 1978).

INVERSION OF ATTENUATION AND DISPERSION

The inverse problem is to solve for wave equation coefficients \( C_1 \) and \( C_2 \) (soil stiffness and damping), given measurements of body wave dispersion and attenuation. The joint inversion of two different data types can be linearized and formulated as an iterative matrix inversion scheme using a Taylor's series expansion limited to the first order terms (Menke 1989). The general form is

\[ G \cdot \Delta m = \Delta d \]  

The Jacobian matrix, \( G \), contains the derivatives of the measurements (phase velocity and attenuation) with respect to the desired parameters (\( C_1 \) and \( C_2 \)). The vector, \( \Delta m \), contains changes to the soil parameters, which reduces the least square error between the observed and predicted measurements. The vector, \( \Delta d \), contains the differences between the observed and calculated measurements.

For this specific problem, the matrix equation (21) is partitioned as follows:

\[
\begin{bmatrix}
\frac{\partial c}{\partial C_1} & \frac{\partial c}{\partial C_2} \\
\frac{\partial \alpha}{\partial C_1} & \frac{\partial \alpha}{\partial C_2}
\end{bmatrix}
\begin{bmatrix}
\Delta C_1 \\
\Delta C_2
\end{bmatrix}
= 
\begin{bmatrix}
\frac{\partial c}{\partial \alpha} \\
\frac{\partial \alpha}{\partial \alpha}
\end{bmatrix}
\]

The upper partition of \( G \) contains the derivatives of phase velocity, \( c \), with respect to the unknown soil parameters. Similarly, the lower partition contains the decay derivatives. Each row of (22) corresponds to a different sampled frequency. The seismic signals are filtered with narrow band-pass filters to observe the frequency dependent behavior of velocity and decay. Each filtered version of the waveforms contributes two rows to the matrix, one for velocity and the other for decay.

The procedure is iterative, beginning with an initial guess for values of \( C_1 \) and \( C_2 \). Solution of (22) leads to sensible changes in \( C_1 \) and \( C_2 \) that reduce the difference between the observed measurements and calculated predictions from (13) and (14).

In this joint inversion, we have two different types of measurement units in the vector, \( \Delta d \). These are velocity \( (m/s) \) and decay \( (1/m) \) at each sampled frequency. This requires a weighting scheme to properly balance the influence of each type of measurement magnitude on the final answer. Since the numerical magnitude of the data values is dependent on the units, one must weight the attenuation measurements (values in the range of tenths) to achieve parity with the velocity measurements (values in the hundreds).

The writer uses a combination of both row and column weighting when inverting (22). Rows within a partition are weighted to compensate for the difference in data units. This basic block weighting of the rows may be further modulated by the reciprocal of the standard deviation for each measurement of velocity and attenuation. Thus, the best data (least uncertainty, small standard deviation) are given greater weight in achieving the solution. Estimates of standard deviation (error bars) for each data type are available from the many different geophone locations within a soil layer which redundantly contribute to measurement of velocity and decay.

Column weighting is done to improve the numerical stability of the calculation. The reciprocal of the maximum derivative in each column is used for the column weighting.

The weighted problem is written as follows:

\[ (W \cdot G \cdot Y) \cdot Y^{-1} \cdot \Delta m = W \cdot \Delta d \]  

where \( W \) is the \((n \times n)\) row weighting matrix, and \( Y \) is the \((2 \times 2)\) column weighting matrix. The diagonal row weighting matrix (for \( n \) frequencies) is given by

\[ W = 
\begin{bmatrix}
w_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & w_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & w_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & w_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & w_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & w_{66}
\end{bmatrix}
\]

where the \( i \)th weights for the upper and lower partition are

\[ w_{ii} = \frac{B \cdot \Delta \alpha \cdot \sigma_{\text{true}}}{\sigma_{\alpha}}, \quad w_{ii} = \frac{(1 - B) \cdot \sigma_{\text{true}}}{\sigma_{\alpha}} \]  

\[(25)\]
Here $B$ is a balance factor that can be used to trim the desired weighting of the relative influence of each data type in achieving a solution to (23). Equal weighting corresponds to a value of 0.5 for $B$, the typical choice. The over-lined variables are the respective mean values of the measured attenuation and velocity. The remaining factors are the standard deviation estimates associated with each measurement, normalized by the minimum standard deviation for each type of data.

The diagonal column weighting matrix is given by

$$ Y = \begin{bmatrix} y_{11} & 0 \\ 0 & y_{22} \end{bmatrix} $$

(26)

where the diagonal elements are equal to the reciprocals of the maximum values found in each column of matrix $G$ respectively.

The least squares solution to (23) is given by Menke (1989):

$$ \Delta m = H^T \cdot \Delta d $$

(27a)

$$ H = Y \cdot \{(WGY)^{-1}\cdot(WGY)^{1\cdot} \cdot W \} $$

(27b)

Here $H$ is the weighted least squares inverse. Typically, the algorithm converges in about 5 iterations.

Once the algorithm has converged on a solution for soil parameters $C_1$ and $C_2$, one must determine corresponding error bars. What is needed is a mapping between the error bars of the calculated quantities ($C_1$ and $C_2$) and the error bars of the measured quantities ($c$ and $\alpha$) at each frequency. Inverse theory provides a solution to this problem (Menke 1989).

Error bars for the wave equation coefficients (soil stiffness and damping) are the diagonal elements of the matrix, $C_m$, given by

$$ C_m = H \cdot C_A \cdot H^T $$

(28)

where $C_A$ is the measurement covariance matrix. For uncorrelated measurement errors, $C_A$ is diagonal, and the nonzero elements are the squares of the standard deviations for each measurement. The square of each measurement standard deviation is the estimate of the variance about the measurement value.

**MEASUREMENT OF DISPERSION AND ATTENUATION**

Ideally, the measurement of velocity dispersion should be done by a robust method which also provides an estimate of the uncertainty of each measurement. Methods based on only two receiver positions (such as the cross-spectrum between two signals from different depths) are vulnerable to modulations introduced from reflections and other non-direct wave fields. The multisignal summations conducted in a semblance calculation provide an effective frequency-wavenumber filter that excludes indirect waves. Further, the average property obtained for an interval will include contributions from each sampled position in the interval (rather than relying on only two points). The writer's method is as follows. For a given subsurface interval of interest, different trial velocities are used to adjust the data into alignment. A semblance value (Sheriff 1991) is computed for each trial velocity, and a search performed to find the optimum velocity and corresponding alignment. The semblance value defines an objective function that must be maximized. This is a nonlinear optimization problem. The maximum is found by a golden section subroutine, which is an adaptation to an algorithm found in the work of Press et al. (1989).

Filtering of the data is one way to measure the variation of velocity with frequency. Selection of filter bandwidth requires a compromise between spectral and temporal resolution. The more narrow the bandwidth of the filter, the less will be the temporal resolution, and vice versa. The writer prefers to use an autoregressive, causal band-pass filter with a bandwidth no greater than 2 Hz. The filters introduce phase delays in the data. However, for any given filter, the delay will be the same for all the geophone stations. Thus, the phase delay will have no effect on the measured velocity.

The error bars for velocity at any measured frequency are found by time shifting the data into alignment with the determined velocity. Then, using one signal as a reference, the relative misalignments between the traces are computed by taking the normalized inner product between traces. The more perfect the alignment, the smaller the error bar will be.

The same filters are also used to measure the variation in amplitude decay with frequency. The filtered data are scanned for RMS amplitude of the direct wave as a function of source-receiver distance. The amplitudes are then corrected for spherical divergence spreading loss and a linear fit made on the logarithm of the corrected amplitudes as a function of propagation path length. The slope of the linear fit yields the decay coefficient, $\alpha$. The least square error of the fit is used in the determination of the decay coefficient error bars. The more perfect the fit, the smaller the error bar.

**DEMONSTRATION ON SYNTHETIC DATA**

A synthetic data set simulating a down-hole experiment was generated by Runge-Kutta integration of the finite difference equation. Eq. (1), cast in matrix form, was used to calculate the waveforms. The spatial sampling was 0.2 m and the temporal integration interval was .0001 s. The simulated source had a peak frequency of 50 Hz and a nominal -6 dB bandwidth of 40 Hz. The synthetic data were resampled to 0.0002 s sample interval after generating the waveforms (comparable to the field recordings to be presented later). The 1-D chain of Kelvin-Voigt elements simulated a 40 m medium. Data corresponding to offsets in the range from 6 to 11 m were then selected for analysis (the selection avoided reflections from the end of the modeled medium). The stiffness and damping were set at $C_1 = 160,000$ m$^2$/s$^2$ and $C_2 = 200$ m/s.

Because a 1-D calculation corresponds to a plane wave, a spherical divergence beam spreading decay was manually applied to the data. This was done by scaling the signal amplitudes by the reciprocal of the distance from the source. This 1/R amplitude scaling simulates a 1/R$^2$ decay in power density associated with a spherical wavefront.

Fig. 4 shows several views of the synthetic data. The true amplitude (after spherical divergence added) is shown in Fig. 4(a). The rapid amplitude decay is quite evident. To better show the changes in the propagating wavefront, Fig. 4(b) presents the data with each signal rescaled by the L2 norm. The L2 norm is simply the Euclidean norm. It is found from the root of the sum of the squares of the samples in any given signal. The changes in the wavelet due to dispersion are rather modest given the short window of observation (5 m). The wavelet does change with distance since the different frequencies propagate with different velocities (dispersion).

Clear evidence of dispersion can be seen in Figs. 4(c) and (d). These are the 30 and 90 Hz filtered versions of the synthetic data (2 Hz bandwidth, 14 pole filters). The reader can better observe the different velocity for each frequency by laying a straight edge along a line of consistent phase in Fig. 4(c) and observing the misalignment with the waves in Fig. 4(d), which are faster. The velocity at each frequency corresponds to the slope in time of the wavefront.

An automated method for determining the velocity at each frequency is to apply time shifts to the data for an assumed velocity, and then compute a semblance value to quantify the quality of the trial alignment. Fig. 5 illustrates this trial-and-error procedure of removing travel time with an assumed ve-
\[ C_1 = 160000 \frac{m^2}{s}, \quad C_2 = 200 \frac{m^2}{s} \]

Determination of the exponential amplitude decay constant, \( \alpha \), is done by formulating the problem as a least squares linear regression for each filtered version. At some distance, \( r \), from the source, the Kelvin-Voigt model predicts the wave amplitude,

\[ A = \left( \frac{A_0 r_0}{r} \right) e^{-\alpha(r - r_0)} \tag{29} \]

where \( A_0 \) is the amplitude of the wave at some reference distance from the source, \( r_0 \). Multiplying both sides of (29) by distance, \( r \), and taking the logarithm of both sides leads to an expression for amplitude decay in decibels:

\[ dB = 20 \log_{10} \left( \frac{A r}{A_0 r_0} \right) = (-20 \log_{10} \alpha)(r - r_0) \tag{30a} \]

\[ dB = (-8.686 \alpha)(r - r_0) \tag{30b} \]

FIG. 4. Finite Difference Synthetic Data Used to Test the Software: (a) True Amplitude Display after Addition of Spherical Divergence Effects; (b) Rescaled Data to Permit Viewing of Waveform; (c) Velocity of 30 Hz Filtered Version; (d) 90 Hz Filtered Version

FIG. 5. Synthetic Data Velocity Analysis by Semblance (30 Hz at 6–11 m Depth)

Velocity. This is done for the 30 Hz filtered version. The correct velocity will align the wavefront along a line of constant arrival time. Fig. 5(a) shows that the correct velocity lies between 615 and 363 m/s. The corresponding semblance values are \( S = 0.8169 \) and \( S = 0.8686 \) respectively. Perfect alignment with noise free data should result in a value approaching unity. The golden section search procedure determined an optimum estimate of velocity equal to 418 m/s (semblance = 0.9413).

Fig. 5(b) shows the application of this velocity to align the data. This procedure is performed for each filtered version of the data to measure a dispersion curve.

FIG. 6. Measurement of Synthetic Data Amplitude Decay (30 Hz)

Finite Difference Data
Synthetic: \( C_1 = 160000 \frac{m^2}{s}, \quad C_2 = 200 \frac{m^2}{s} \)
Solution: \( C_1 = 166282 \frac{m^2}{s}, \quad C_2 = 208 \frac{m^2}{s} \)

FIG. 7. (a) Measured Velocity Dispersion of Synthetic Data; (b) Amplitude Decay of Synthetic Data
Thus, the slope of a line on a decibel vs. distance \((r - r_0)\) plot is \((-8.686)\). Fig. 6 shows such a plot for the synthetic data. The decibel (logarithmic) scale linearizes the exponential decay. The value of \(\alpha(1/m)\) is easily computed from the slope (dB/m) by dividing by \(-8.686\).

Fig. 7 shows the measurements of decay and dispersion as circles with error bars (95% confidence interval). The calculated dispersion and decay from the solution for \(C_1\) and \(C_2\) is displayed as a solid curve through the measured points. The solution is quite close to the values of \(C_1\) and \(C_2\) used to generate the data. The somewhat larger stiffness and damping values result from the additional computational dispersion which is common in finite difference methods. The effect is due to the finite discretization of the computational problem. Mechanical waves propagating through discrete lattice structures also exhibit this type of additional dispersion.

**DEMONSTRATION OF FIELD DATA**

Two different borehole surveys have been selected to demonstrate the writer’s method. One borehole was located in Logan, Utah, and the other in Boise, Idaho. The Utah data were acquired at the GeoLogan97 field day site. The survey was conducted on July 15, 1997, during the first meeting of the ASCE GeoInstitute. This site exhibits low levels of viscous damping in a soil profile consisting of fine grained sands, silts, and clays.

The Idaho data were acquired in coarse grained granular soils consisting of sands, gravels and cobbles. The Idaho data were collected between episodes of sparging that were being conducted to treat contaminated ground water. The levels of viscous damping are very large at this location.

**Utah Case History**

Fig. 8 shows the SH-wave data from the horizontal components. Following hodogram analysis for tool orientation, the data were rotated parallel to the source polarization. The signals have been scaled to remove the amplitude decay with depth, permitting the reader to better observe the waveform of the direct arriving wave. Also shown in Fig. 8 is the soil behavior type classification from a neighboring cone penetrometer (CPT-3) survey conducted by ConeTec in November 1996. The CPT survey was done in preparation for the GeoLogan97 meeting.

The change in slope of the first arrivals at the top of the sand clearly indicates a change in wave velocity. The writer chose to apply the method to intervals both above and below this change in velocity. One interval (at 2 to 7 m depth) is in the low velocity saturated silts and clays. The other interval (at 8 to 13 m depth) is in the upper portion of the higher velocity sand. Thus each interval is about 5 m thick and includes about 20 geophone stations (0.25 m station spacing).

The source was the one shown in Fig. 2(b). The entire data set was acquired by the writer without any helper. Working alone, it took 3 hours and 15 minutes to acquire the data. With a helper, this would have been reduced to about 2 hours. The data were collected on a Bison 9048 engineering seismograph with a 0.00025 second sample interval and filters set to 4 and 1,000 Hz.

Fig. 9 shows the dispersion and decay for the silt interval. In this and all later presentations, error bars are for 95% confidence limits (random error). Bias will always be present and is difficult to quantify. Bias is introduced by choices, such as the precise limits on the interval, the filter frequencies, and the weighting scheme used in the inversion (all of which affect the resulting \(C_1\) and \(C_2\) determinations).

Clearly, the lack of velocity dispersion and nearly flat am-

![FIG. 8. Down-Hole SH-Wave Field Data from Logan, Utah, Site](image)

Solution:

\[ C_1 = 25567 \pm 218 \]

\[ C_2 = 1 \pm 4 \]

![FIG. 9. GeoLogan SH-Wave Data (2-7 m): (a) Measured Velocity Dispersion from Silt Interval in Utah Data; (b) Amplitude Decay from Silt Interval in Utah Data](image)
plitude decay response with frequency suggest a low level of damping. This low level of damping is further supported by the need to reduce the recording instrument preamplifier gain by 20 dB from the normal setting used in Idaho, where signal losses have been greater. The smooth curves in Fig. 9 are the dispersion and amplitude decay computed from the solution:

\[ C_1 = 25,567 \pm 218 \text{ m}^2/\text{s}^2 \text{ stiffness} \]

\[ C_2 = 1 \pm 1 \text{ m}^2/\text{s} \text{ damping} \]

The relaxation time for the silt (from the ratio of \( C_2/C_1 \)) is only 39 \pm 29 \mu s. Such a short relaxation time, uncertain as it may be, could be an indication that the pore fluids are moving with the frame in this frequency band (presumably due to the low level of permeability that one would associate with finer grained materials). If the fluids are moving with the frame, rather than through it, there would be less viscous drag.

Fig. 10 shows the dispersion and decay measurements for the sand interval (at 8 to 13 m depth). The velocity dispersion is slightly greater than in the silt. The variation of amplitude decay with frequency is significantly greater than for the silt. Again, the smooth curves are computed from the solution:

\[ C_1 = 51,343 \pm 375 \text{ m}^2/\text{s}^2 \text{ stiffness} \]

\[ C_2 = 14 \pm 1 \text{ m}^2/\text{s} \text{ damping} \]

Damping has increased by a factor of 14, and the stiffness has doubled. For the sand, the relaxation time is computed to be 273 \pm 19 \mu s. Clearly, the role of viscous damping has over-

\[ \text{Solution: } C_1 = 51,343 \pm 375 \text{ m}^2/\text{s}^2 \]

\[ C_2 = 14 \pm 1 \text{ m}^2/\text{s} \]

whelmed the increase in stiffness. It may be that the presumed increase in permeability which one would likely associate with a larger effective grain size is partly responsible for the increased damping. That is, the sandy soil may be of sufficient permeability to afford more fluid-frame interaction. The result would be more viscous damping in the sand than in the silt.

**Idaho Case History**

Fig. 11 shows the Idaho SH-wave data rotated into alignment with the source polarization. Also shown is a description of the material observed during the drilling process. The sudden change in slope of the direct arriving wavefront occurs at the water table, suggesting that the SH-wave speed increases below the vadose zone.

The data were acquired with the horizontal impact source shown in Fig. 2(a). Data collection was on December 20, 1994. Downhole stations were acquired every 0.5 m. The recording instrument was a Bison 9048 engineering Seismograph with a sample interval of 0.0002 s and filter set at 8 and 1,000 Hz. With one helper, the survey took 1 hour 40 minutes for data collection.

Stiffness and damping for two intervals are presented here. A shallow interval (just below the water table, at 5 to 10 m) was found to exhibit extreme damping. The other deeper interval (at 10 to 15 m) was found to have significantly less damping. Since the wave velocity at the dominant frequency of about 50 Hz is not much different in the two zones, an elastic analysis would fail to detect a significant difference between the two intervals. As will be shown, the difference becomes evident only when one examines velocity and amplitude decay as a function of frequency.

Fig. 12 shows the measured velocity dispersion and attenuation for the upper interval. The smooth curve is calculated from the inversion solution:

\[ C_1 = 94,917 \pm 2,913 \text{ m}^2/\text{s}^2 \text{ stiffness} \]

\[ C_2 = 255 \pm 9 \text{ m}^2/\text{s} \text{ damping} \]

The corresponding relaxation time is 2,686 \pm 125 microseconds. It is evident that extremely large viscous forces are causing the dispersion and amplitude decay. The use of a Kelvin model is well justified since the dispersion and decay observed agree well with the model (see Eqs. [12], [13], and
In the language of inverse theory, the error bars in the solution are chiefly a result of errors in the measured quantities (data error) rather than due to the choice of an inappropriate model (resolving error).

One possible contributing factor to the large damping may be residual trapped air in the pores. The survey was conducted four and a half months following the termination of a seven-month sparging-ground-water treatment program. At this point, there is no way to know if trapped air was present during the survey. Trapped air would reduce the degree of water saturation. Significant alteration of SH-wave velocity has been documented in partially saturated soils (Wu 1984). In any case, the granular nature of the soils would predict sufficient permeability for interaction between fluids and the soil frame, the result being viscous damping.

Fig. 13 shows the measurements of dispersion and decay for the deeper interval. It is clear that significantly less viscous damping is present from these data. The smooth curves were calculated from the solution:

\[
C_1 = 182,751 \pm 4,860 \text{ m}^2/\text{s}^2 \text{ stiffness}
\]
\[
C_2 = 69 \pm 17 \text{ m}^2/\text{s} \text{ damping}
\]

The relaxation time for this interval is computed to be 378 ± 94 μs. In comparing the two intervals, it is clear that the lower interval is stiffer with less damping. The fact that damping can raise the velocity of the wave is easily overlooked if not measured. This may be important when computing other quantities such as shear modulus from velocity alone.

CONCLUSIONS

A method for determining in-situ stiffness and damping has been presented. The method is to jointly invert both measurements of SH-wave velocity dispersion and spatial amplitude decay, corrected for beam divergence. The method is consistent with current engineering practice and uses a Kelvin-Voigt constitutive model. The field examples presented here demonstrate that the method works in practice. Furthermore, the current use of the Kelvin-Voigt model in engineering (where pore water is present) is supported by these in situ determinations.

Since damping will increase the wave velocity, it is possible to introduce significant errors by computing shear modulus from wave velocity alone. It has been common practice to compute the shear modulus from measurements of the dominant group velocity of SH-waves. This is like measuring only the resonant frequency in a spring/mass/dashpot experiment, and then computing the spring constant without consideration of any damping effects.

The actual measurement of amplitude decay is essential if damping is to be determined. To measure amplitude decay, the writer's method invokes a significant amount of redundancy and averaging to overcome variations in borehole to formation coupling. This strategy also helps reduce the effects of constructive and destructive interference presented from scattered and reflected waves. Further, it is this redundancy which also permits an estimate of the errors involved.

Finally, the writer would like to speculate that his often observed increase in shear wave velocity at the water table
may be an indication of a shift in dominance from contact (grain-grain) friction in the vadose zone to viscous (fluid-frame) friction below the water table. It is likely that both relaxation mechanisms exist at all times, but the high viscosity of water (relative to air) may shift the balance to a viscoelastic model below the water table. Evidence for this conjecture may also be found in resonant column studies of saturated and dry soils (Stoll 1985).

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APPENDIX. REFERENCES


