Short Note
Surface-Wave Green’s Tensors in the Near Field
by Matthew M. Haney and Hisashi Nakahara

Abstract We demonstrate the connection between theoretical expressions for the correlation of ambient noise Rayleigh and Love waves and the exact surface-wave Green’s tensors for a point force. The surface-wave Green’s tensors are well known in the far-field limit. On the other hand, the imaginary part of the exact Green’s tensors, including near-field effects, arises in correlation techniques such as the spatial autocorrelation (SPAC) method. Using the imaginary part of the exact Green’s tensors from the SPAC method, we find the associated real part using the Kramers–Kronig relations. The application of the Kramers–Kronig relations is not straightforward, however, because the causality properties of the different tensor components vary. In addition to the Green’s tensors for a point force, we also derive expressions for a general point moment tensor source.

Introduction
Recently, Haney et al. (2012) extended the spatial autocorrelation (SPAC) method of Aki (1957) to the full tensor of possible three-component correlations, including correlations of mixed components. The complete Rayleigh-ϕR and Love-ϕL wave correlation coefficient tensors are

\[ \phi^R(r, \omega) = \begin{bmatrix} \phi_{ZZ}^R & \phi_{ZR}^R & \phi_{ZT}^R \\ \phi_{ZR}^R & \phi_{RR}^R & \phi_{RT}^R \\ \phi_{ZT}^R & \phi_{RT}^R & \phi_{TT}^R \end{bmatrix} = P^R(\omega) \]

\[ \times \begin{bmatrix} J_0(\omega r/c_R) & -R J_1(\omega r/c_R) & 0 \\ R J_1(\omega r/c_R) & R^2[J_0(\omega r/c_R) - J_2(\omega r/c_R)]/2 & 0 \\ 0 & 0 & R^2[J_0(\omega r/c_R) + J_2(\omega r/c_R)]/2 \end{bmatrix} \]

(1)

and

\[ \phi^L(r, \omega) = \begin{bmatrix} \phi_{ZZ}^L & \phi_{ZR}^L & \phi_{ZT}^L \\ \phi_{ZR}^L & \phi_{RR}^L & \phi_{RT}^L \\ \phi_{ZT}^L & \phi_{RT}^L & \phi_{TT}^L \end{bmatrix} = P^L(\omega) \]

\[ \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & [J_0(\omega r/c_L) + J_2(\omega r/c_L)]/2 & 0 \\ 0 & 0 & [J_0(\omega r/c_L) - J_2(\omega r/c_L)]/2 \end{bmatrix}, \]

(2)

in which \( \omega \) is angular frequency, \( r \) is radial distance, \( c_R \) is Rayleigh-wave velocity, \( c_L \) is Love-wave velocity, \( R \) is the ratio of the horizontal-to-vertical motion of the Rayleigh waves, \( P^R \) is the power spectrum of the Rayleigh waves, \( P^L \) is the power spectrum of the Love waves, and \( J_0, J_1, \) and \( J_2 \) are Bessel functions of the zero, first, and second orders, respectively.

As discussed in Nakahara (2006), correlations of scalar waves are related to the imaginary part of the scalar-wave Green’s function. This property of correlations has been discussed extensively in the literature (Sánchez-Sesma and Campillo, 2006; Snieder et al., 2009). Recently, Haney et al. (2012) made the ansatz that the imaginary part of the surface-wave Green’s tensors should be similarly related to the tensors arising from the correlation of multicomponent
surface-wave data, equations (1) and (2). Based on this ansatz, we use the Kramers–Kronig relations to recover the associated real parts from the imaginary parts and find closed-form expressions for the exact surface-wave Green’s tensors. As expected, the real part is singular at the origin (Snieder et al., 2009). The surface-wave Green’s tensors are well known in the far-field limit, as shown by Aki and Richards (1980) in their equations 7.145 and 7.146. Aki and Richards (1980) also show an expression that can be used to obtain the exact Love-wave Green’s tensor in equation 7.141; however, it is not as clearly presented as the far-field formula. Moreover, a companion formula for Rayleigh waves is not presented. To our knowledge, the only other presentation of exact surface-wave Green’s tensors in the geophysical literature is given in spherical (global) coordinates by Dahlen and Tromp (1998) in their equation 11.20. Although the equation is correct, the connection to the expressions in the Cartesian coordinate system of Aki and Richards (1980) is not clear. Here, we give exact surface-wave Green’s tensors in Cartesian coordinates that represent the extension of equations 7.145 and 7.146 in Aki and Richards (1980) to include the near field. The Love-wave Green’s tensor is given in a clear and compact format, in contrast to equation 7.141 of Aki and Richards (1980). In spite of the obscure notation used in equation 7.141 of Aki and Richards (1980), we are able to compare it with our result for the Love-wave Green’s tensor and verify the two expressions are in agreement. This formally demonstrates the connection between the SPAC method (Aki, 1957) and the surface-wave Green’s tensors.

From equations (1) and (2) and knowledge of cylindrical harmonics, one may intuitively guess that the real part of the Green’s tensor is related to Bessel functions of the second kind, or Neumann functions. By using the Kramers–Kronig relations, we rigorously show that this is true. The details of the derivation are given in Appendices A–D and only the main results for the Green’s tensors and the solution due to a general point moment tensor are given in the text. We also demonstrate the unusual properties of the exact Green’s tensors in the near field by considering Love-wave radiation from an explosive source. Although the distinction between body and surface waves is physically meaningless in the near field (Snieder, 2002), the expressions we derive should provide a good description of near-field seismic radiation for sources that primarily radiate surface waves, that is, sources located at or near the Earth’s surface. In addition, the exact surface-wave Green’s tensors can be used to construct virtual earthquakes from ambient noise correlations (Denolle et al., 2013) when the interstation distance is on the order of a wavelength or less. The expressions constitute the link between the SPAC method and the surface-wave Green’s function and may prove useful for deriving the spatial correlation between other measurable seismic quantities, such as rotation and strain.

### Exact Rayleigh- and Love-Wave Green’s Tensors

In equations (1) and (2), we utilize an ordering convention for the columns and rows of the correlation tensors given by $(Z, R, T)$. Aki and Richards (1980) present surface-wave Green’s tensors in a Cartesian coordinate system using the convention $(N, E, Z)$. The mapping for the Green’s tensor $G_{ij}$ between the two coordinate systems is given by

\[
\begin{bmatrix}
G_{NN} & G_{NE} & G_{NZ} \\
G_{EN} & G_{EE} & G_{EZ} \\
G_{ZN} & G_{ZE} & G_{ZZ}
\end{bmatrix}
= \begin{bmatrix}
0 & \cos \phi & -\sin \phi \\
\sin \phi & 0 & \cos \phi \\
\cos \phi & \sin \phi & 0
\end{bmatrix}
\times \begin{bmatrix}
G_{ZZ} & G_{ZR} & G_{ZT} \\
G_{RZ} & G_{RR} & G_{RT} \\
G_{TZ} & G_{TR} & G_{TT}
\end{bmatrix}
\times \begin{bmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0
\end{bmatrix},
\]

in which $\phi$ is the azimuthal angle. We use $N, E,$ and $Z$ coordinates in the following to facilitate comparison with expressions from Aki and Richards (1980). In addition, we attempt to obey all other conventions of Aki and Richards (1980) as closely as possible. Note that the azimuthal angle $\phi$ should not be confused with the Rayleigh- $\phi^R$ and Love- $\phi^L$ wave correlation coefficient tensors. Expressions for $G_{ij}$ in the $Z, R,$ and $T$ coordinate system for Rayleigh and Love waves are derived in Appendices A–D.

All of the equations we present can be demonstrated to converge to the expressions of Aki and Richards (1980) in the far-field limit. The generalization of equations 7.145 and 7.146 from Aki and Richards (1980), or equations 7.146 and 7.147 from Aki and Richards (2002), are given by

\[
G_{\text{LOVE}} = \sum_n i l_1(z) l_1(h) \frac{8c U l_1}{h} \times \begin{bmatrix}
\frac{1}{2} [H_0^{(1)}(k_\pi r) + H_2^{(1)}(k_\pi r) \cos 2\phi] & H_2^{(1)}(k_\pi r) \sin \phi \cos \phi & 0 \\
H_2^{(1)}(k_\pi r) \sin \phi \cos \phi & \frac{1}{2} [H_0^{(1)}(k_\pi r) - H_2^{(1)}(k_\pi r) \cos 2\phi] & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
G_{NN} & G_{NE} & G_{NZ} \\
G_{EN} & G_{EE} & G_{EZ} \\
G_{ZN} & G_{ZE} & G_{ZZ}
\end{bmatrix}
= \begin{bmatrix}
0 & \cos \phi & -\sin \phi \\
\sin \phi & 0 & \cos \phi \\
\cos \phi & \sin \phi & 0
\end{bmatrix}
\times \begin{bmatrix}
G_{ZZ} & G_{ZR} & G_{ZT} \\
G_{RZ} & G_{RR} & G_{RT} \\
G_{TZ} & G_{TR} & G_{TT}
\end{bmatrix}
\times \begin{bmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0
\end{bmatrix},
\]

in which $\phi$ is the azimuthal angle. We use $N, E,$ and $Z$ coordinates in the following to facilitate comparison with expressions from Aki and Richards (1980). In addition, we attempt to obey all other conventions of Aki and Richards (1980) as closely as possible. Note that the azimuthal angle $\phi$ should not be confused with the Rayleigh- $\phi^R$ and Love- $\phi^L$ wave correlation coefficient tensors. Expressions for $G_{ij}$ in the $Z, R,$ and $T$ coordinate system for Rayleigh and Love waves are derived in Appendices A–D.

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\[
G_{\text{LOVE}} = \sum_n i l_1(z) l_1(h) \frac{8c U l_1}{h} \times \begin{bmatrix}
\frac{1}{2} [H_0^{(1)}(k_\pi r) + H_2^{(1)}(k_\pi r) \cos 2\phi] & H_2^{(1)}(k_\pi r) \sin \phi \cos \phi & 0 \\
H_2^{(1)}(k_\pi r) \sin \phi \cos \phi & \frac{1}{2} [H_0^{(1)}(k_\pi r) - H_2^{(1)}(k_\pi r) \cos 2\phi] & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
\[ G_{\text{RAYLEIGH}} = \sum_n \frac{i}{8cU_1} \left[ \left( \frac{r_1(z)r_2(h)}{2} [H_0^{(1)}(k_n r) - H_2^{(1)}(k_n r) \cos 2\phi] - r_1(z) r_1(h) H_2^{(1)}(k_n r) \sin \phi \cos \phi \right) r_1(z) r_2(h) H_1^{(1)}(k_n r) \right] \times \left[ -r_2(z) r_1(h) H_1^{(1)}(k_n r) \cos \phi \right]. \] (5)

in which \( k_n = |\omega|/c_n \) with \( c_n \) equal to either the Love- or Rayleigh-wave phase velocity of the \( n \)th mode and \( H_0^{(1)} \), \( H_1^{(1)} \), and \( H_2^{(1)} \) are Hankel functions of the first kind of zero, first, and second orders, respectively. The above expressions are valid for positive frequencies, whereas for negative frequencies the Hankel functions of the first kind are replaced by the negative Hankel functions of the second kind, that is \( H_0^{(1)}(k_n r) \leftrightarrow -H_2^{(2)}(k_n r) \). This substitution for negative frequencies holds for all of the following expressions as well. In accordance with Aki and Richards (1980), the dependence on the mode number \( n \) for the phase velocity \( c \), group velocity \( U \), integral \( I_1 \), and eigenfunctions \( l_1, r_1 \), and \( r_2 \) is suppressed in the above equations for brevity, although the dependence is implied.

We note the Love-wave Green’s tensor (equation (4)) can be derived from equation 7.141 in Aki and Richards (1980). This represents an alternative way of deriving the exact Love-wave Green’s tensor and an independent check on equation (4). A similar expression for Rayleigh waves is not given in Aki and Richards (1980). We have performed this check for Love waves and found agreement between the two results. We do not show the complete process in detail but describe the steps required to modify equation 7.141 in Aki and Richards (1980) for comparison to equation (4). Equations 7.113, 7.117, and 7.125 in Aki and Richards (1980) and the text prior to 7.138 in Aki and Richards (1980) regarding the replacement of \( J_0 \) by \( H_0^{(1)}/2 \) are all needed to rewrite equation 7.141 in a form suitable for comparison. To give some insight into the process, we mention the expression for \( T_{\text{LOVE}}^x \) in equation 7.117 of Aki and Richards (1980) contains both an angle and a radial derivative. The angle derivative gives the combination \( (J_0 + J_2) \) by one recursive property of Bessel functions and the radial derivative gives \( (J_0 - J_2) \) by another recursive property. These two combinations appear as the RR and TT entries of the Love-wave correlation tensor shown in equation (2). The connection between the Love-wave correlation tensor and the exact Love-wave Green’s tensor is discussed in Appendix D.

Following Aki and Richards (1980), we can also derive the expressions for Love and Rayleigh waves from a general point moment tensor source using the Green’s tensors. For Love waves, the \( u_x \) component of motion is given by

\[ u_x^{\text{LOVE}} = \sum_n \frac{i l_1(z)}{8cU_1} \left[ \left( \frac{l_1(h)}{2} \left( (M_{xx} \cos \phi + M_{xy} \sin \phi) H_1^{(1)}(k_n r) \right) \right) \times \left( \cos 2\phi \left[ H_1^{(1)}(k_n r) - H_3^{(1)}(k_n r) \right] \right) - (M_{xx} \cos \phi \right] + \left( M_{xy} \sin \phi \right) H_1^{(1)}(k_n r) \times \left[ \left( \frac{M_{zz}}{2} [H_0^{(1)}(k_n r) - H_2^{(1)}(k_n r) \cos 2\phi] \right) \right] + \left( M_{yz} H_2^{(1)}(k_n r) \sin \phi \cos \phi \right). \] (6)

which is the generalization of equation 7.147 from Aki and Richards (1980), or equation 7.148 from Aki and Richards (2002). The \( u_z \) component is given by

\[ u_z^{\text{LOVE}} = \sum_n \frac{i l_1(z)}{8cU_1} \left[ \left( \frac{l_1(h)}{2} \left( (M_{xx} \cos \phi + M_{xy} \sin \phi) H_1^{(1)}(k_n r) \right) \right) \times \left( \cos 2\phi \left[ H_1^{(1)}(k_n r) - H_3^{(1)}(k_n r) \right] \right) + (M_{xx} \cos \phi \right] + \left( M_{xy} \sin \phi \right) H_1^{(1)}(k_n r) \times \left[ \left( \frac{M_{zz}}{2} [H_0^{(1)}(k_n r) - H_2^{(1)}(k_n r) \cos 2\phi] \right) \right] + \left( M_{yz} H_2^{(1)}(k_n r) \sin \phi \cos \phi \right). \] (7)

We emphasize that both \( u_x \) and \( u_z \) need to be specified because Love waves are not strictly transverse in the near field.

For Rayleigh-wave wave from a point moment tensor source, the \( u_z \) component of motion is given by
which is the generalization of equation 7.149 from Aki and Richards (2002). The $u_x$ component of motion is given by

$$u_{x,\text{RAYLEIGH}} = \sum_n \frac{i r_1(z)}{8 \pi U I_1} \times \left[ r_1(h) k_n \frac{1}{2} \left( M_{xx} \cos \phi \right) 
+ M_{yy} \sin \phi \left\{ H_1^{(1)}(k_n r) \right. 
- \cos \frac{2 \phi}{2} [H_1^{(1)}(k_n r) - H_3^{(1)}(k_n r) \left. \right] 
+ (M_{xx} \cos \phi + M_{yy} \sin \phi) [H_1^{(1)}(k_n r) 
- H_3^{(1)}(k_n r) \cos 2 \phi] 
+ r_2(h) k_n [H_0^{(1)}(k_n r) - H_2^{(1)}(k_n r)] (M_{zz} \cos \phi 
+ M_{yz} \sin \phi) \right. 
\left. + \frac{dr_1}{dz} \left( \frac{M_{yz} [H_0^{(1)}(k_n r)}{2} \right) \frac{dz}{h} \right] H_1^{(1)}(k_n r) M_{zz} \cos \phi \right], \quad (8)$$

with $u_y$ given by

$$u_{y,\text{RAYLEIGH}} = \sum_n \frac{i r_1(z)}{8 \pi U I_1} \times \left[ r_1(h) k_n \frac{1}{2} \left( M_{xx} \cos \phi \right) 
+ M_{yy} \sin \phi \left\{ H_1^{(1)}(k_n r) \right. 
- \cos \frac{2 \phi}{2} [H_1^{(1)}(k_n r) - H_3^{(1)}(k_n r) \left. \right] 
+ (M_{xx} \cos \phi + M_{yy} \sin \phi) [H_1^{(1)}(k_n r) 
- H_3^{(1)}(k_n r) \cos 2 \phi] 
+ r_2(h) k_n [H_0^{(1)}(k_n r) - H_2^{(1)}(k_n r)] (M_{zz} \cos \phi 
+ M_{yz} \sin \phi) \right. 
\left. + \frac{dr_2}{dz} \left( \frac{M_{yz} [H_0^{(1)}(k_n r)}{2} \right) \frac{dz}{h} \right] H_2^{(1)}(k_n r) M_{xx} \cos \phi \right], \quad (9)$$

As was the case with Love waves, both $u_x$ and $u_y$ are needed because Rayleigh waves are not strictly radial in the near field.

**Example**

As an example of the unusual properties of the exact Green’s tensors in the near field, consider the case of Love waves due to an explosion. In this case, the particle motion is purely radial and given by

$$u_{x,\text{RAYLEIGH}} = \left( \cos \phi, \sin \phi \right) \sum_n \frac{i r_1(z) I_1(h) k_n}{32 \pi U I_1} \left[ H_1^{(1)}(k_n r) \right] 
+ H_3^{(1)}(k_n r) \right] \quad (10)$$

For large arguments, $H_1^{(1)} = -H_3^{(1)}$ and Love-wave radiation is zero in the far field, as expected for an explosion. However, in the near field, Love waves are nonzero and are radially polarized.

**Conclusion**

We have exploited previous results from the SPAC method and the Kramers–Kronig relations to derive exact surface-wave Green’s tensors valid in the near field. This demonstrates the deep connection between the SPAC method and the surface-wave Green’s tensors. The expressions should be useful for describing near-source wavefields when surface waves are the dominant wave type. Because surface waves comprise the majority of microseismic noise, these expressions can be used to calculate accurate virtual earthquakes from ambient noise correlations between two seismometers at close range.

**Data and Resources**

No data were used in this paper.

**Acknowledgments**

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**References**


### Appendix A

**Kramers–Kronig Relations**

We use the following convention for the Fourier transform:

\[
S(\omega) = \int_{-\infty}^{\infty} s(t) \exp(i\omega t) dt = F[s(t)]
\]

(A1)

and

\[
s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \exp(-i\omega t) d\omega = F^{-1}[S(\omega)],
\]

(A2)

in which \(s(t)\) is a time series, \(S(\omega)\) is its Fourier transform, and the symbols \(F\) and \(F^{-1}\) are shorthand for the forward and inverse Fourier transforms, respectively. The following are Fourier transform pairs for derivatives in both domains:

\[
\frac{dS}{d\omega} = F[is\omega(t)]
\]

(A3)

and

\[
\frac{ds}{dt} = F^{-1}[-i\omega S(\omega)],
\]

(A4)

in which the sign on the right sides of equations (A3) and (A4) changes according to the convention in equations (A1) and (A2). Our convention for Hilbert transforms in both domains follows from the above expressions for derivatives:

\[
H[S] = F[\text{sgn}(\omega)S(\omega)],
\]

(A6)

in which \(H\) represents the Hilbert transform. As a result, the convention for the Hilbert transforms in both domains is linked to the Fourier transform convention. The derivative and Hilbert transform of a time series, as shown in equations (A4) and (A6), have identical phase spectra. This type of property exists between the dual relationships in equations (A3) and (A5) as well.

Given these conventions, we find the associated Kramers–Kronig relations for a causal time series. A generic form for a causal time series \(s(t)\) is

\[
s(t) = z(t) + \text{sgn}(t)z(t),
\]

(A7)

in which \(z(t)\) is a time series that is symmetric in time, that is \(z(t) = z(-t)\). We require \(\text{sgn}(t)\) to be symmetric so that its Fourier transform is real valued. Taking the Fourier transform of equation (A7) gives

\[
F[s(t)] = F[z(t)] + F[\text{sgn}(t)z(t)]
\]

\[
= F[z(t)] - iF[\text{sgn}(t)z(t)],
\]

(A8)

in which, as stated previously, \(F[z(t)]\) is purely real because \(z(t) = z(-t)\). Using the notation \(S = F[s(t)]\) and \(Z = F[z(t)]\) and equation (A5), equation (A8) can be rewritten as

\[
S = Z - iH[Z],
\]

(A9)

from which we can write the following Kramers–Kronig relations:

\[
H[\text{Re}(S)] = -\text{Im}(S)
\]

(A10)

and

\[
\text{Re}(S) = H[\text{Im}(S)].
\]

(A11)

Equation (A11) can be derived from equation (A10) because the inverse Hilbert transform is the negative of the forward transform: \(H^{-1} = -H\).

In this paper, we use the term causal to mean any time series that satisfies the above Kramers–Kronig relations. Such a time series is one sided because it is equal to zero for times prior to \(t = 0\). By causal, we do not mean the time series has the related minimum phase property. As described in Claerbout (1976), the collection of time series with the minimum phase property comprises a subset of all possible one-sided time series. If \(s(t)\) were a minimum phase time series, then, in addition to the Kramers–Kronig relations, it would also satisfy the following relations:

\[
H[\log A] = -\theta
\]

(A12)
\[ \log A = H[\theta], \]  
(A13)
in which \( \theta = \tan^{-1}[\text{Im}(S)/\text{Re}(S)] \) is the phase spectrum and 
\( A = \sqrt{\text{Re}(S)^2 + \text{Im}(S)^2} \) is the amplitude spectrum. Note the 
minimum phase conditions and the Kramers–Kronig relations have a similar form 
because either the logarithmic amplitude and phase spectra or the real and imaginary 
parts of the complex spectra are Hilbert transforms of each other, respectively. 
Within the context of this paper, we define causality by the Kramers–Kronig 
relations for the Rayleigh wave Green’s function for an isotropic wavefield, one 
in which energy propagates equally in all directions. For an isotropic wavefield, 
\( r/c \) is based on the above ansatz and its agreement with equation 7.141 of Aki and Richards (1980) provides independent support for equation (B1). However, it remains to be proven mathematically that equation (B1) is correct in the near field and for this reason we refer to it as an ansatz.

From Haney et al. (2012), the time-domain expression for the Hilbert transform of the ZZ correlation is given for times prior to the direct arrival \(|t| < r/c\) as

\[ H[\phi_{ZZ}(r, t)] = \frac{2e^{-m\gamma_m}}{\pi r} \sum_{m=1}^{\infty} \text{Re}[\gamma_m] U_{m-1}(\frac{ct}{r}), \]  
(B4)
in which, in the above expression, \( U_{m-1}(\frac{ct}{r}) \) is the \((m-1)\) order Chebyshev polynomial of the second kind and \( \gamma_m \) is a measure of the distribution of energy propagation as a function of direction. We focus on times prior to the direct arrival because a causal propagating waveform will be equal to zero for \(|t| < r/c\). The relationships between the correlation tensor and Green’s tensor in equations (B1), (B2), and (B3) hold for an isotropic wavefield, one in which energy propagates equally in all directions. For an isotropic wavefield, \( \gamma_0 = 1 \) and \( \gamma_m = 0 \) for \( m \neq 0 \) for the ZZ component (Haney et al., 2012). From equation (B4), this means that \( H[\phi_{ZZ}(r, t)] \) equals 0 for times prior to the direct arrival \(|t| < r/c\). Thus, \( H[\phi_{ZZ}(r, t)] \) is the sum of a causal propagating waveform and a time-reversed version of a causal propagating waveform. From equation (B1), the Hilbert transform of \( \phi_{ZZ} \) is related to the sum of \( G_{ZZ} \) and the negative of the time reverse of \( G_{ZZ} \). Therefore, \( G_{ZZ} \) must be causal and, as a result, the Kramers–Kronig relations can be applied to it.

From equations (1) and (B3), we know that

\[ \text{Im}(G_{ZZ}) = \frac{\text{sgn}(\omega)}{4} J_0(\omega r/c), \]  
(B5)
in which we have set the spectral power in equation (1) to unity: \( P^R(\omega) = 1 \). Based on the Kramers–Kronig relation from equation (A11), we find that

\[ \text{Re}(G_{ZZ}) = \frac{1}{4} H[\text{sgn}(\omega) J_0(\omega r/c)]. \]  
(B6)

We use the following identity (Erdelyi, 1954, p. 254, equation 14 with \( \nu = 0 \)):

\[ H[\text{sgn}(x) J_0(x)] = -Y_0(|x|), \]  
(B7)
in which \( Y_0 \) is the zeroth-order Bessel function of the second kind, and obtain the following for the real part

\[ \text{Re}(G_{ZZ}) = -\frac{1}{4} Y_0(|\omega| r/c). \]  
(B8)
From equations (B5) and (B8), we obtain the exact form for the ZZ component of the Rayleigh-wave Green’s tensor:

$$G_{ZZ} = \frac{1}{4} [-Y_0(|\omega|r/c) + i\text{sgn}(\omega)J_0(\omega r/c)]. \quad (B9)$$

In contrast to the expression in Aki and Richards (1980), this is valid at all distance ranges including the near field.

Haney et al. (2012) assume a normalization for the Rayleigh waves of \( r_2^2/(2cU_1) = 1 \), in which \( r_2 \) is the value of the vertical component of the Rayleigh-wave eigenfunction at the surface, \( U \) is the group velocity, and \( I_1 \) is an integral over depth of density weighted by the eigenfunctions at depth (Aki and Richards, 1980). Therefore, equation (B9) in its most general form has a factor of \( r_2^2/(2cU_1) \) on the right side. To make the connection with the Green’s function, we generalize the expressions from Haney et al. (2012) to accommodate the case when the two receivers being correlated are at different depths. Thus, the term \( r_2^2 \) becomes \( r_2(z)r_2(h) \), in which \( z \) and \( h \) are the source and receiver depths, respectively. This yields the following general form for the Green’s function:

$$G_{ZZ} = \frac{r_2(z)r_2(h)}{8cU_1} H_0^{(1)}(\omega r/c). \quad (B10)$$

Given the Hankel function of the first kind \( H_0^{(1)} = J_0+iY_0 \), equation (B10) may be written for positive frequencies (\( \omega > 0 \)) as

$$G_{ZZ} = \frac{ir_2(z)r_2(h)}{8cU_1} H_0^{(1)}(\omega r/c). \quad (B11)$$

In addition, given the Hankel function of the second kind \( H_0^{(2)} = J_0-iY_0 \), the corresponding expression for negative frequencies is

$$G_{ZZ} = -\frac{ir_2(z)r_2(h)}{8cU_1} H_0^{(2)}(\omega r/c). \quad (B12)$$

The absolute value of the frequency \(|\omega|\) is needed in equation (B12) because \( Y_0 \) is not an even function, in contrast to \( J_0 \). All equations in this short note apply to negative frequencies by setting \( n \)th order Hankel functions of the first kind to the negative of those of the second kind: \( H_n^{(1)}(\omega r/c) \rightarrow -H_n^{(2)}(\omega r/c) \). Note that the argument of the Hankel functions is in general equal to \( \omega r/c \), because it is \( \omega r/c \) for positive frequencies and \( |\omega| r/c \) for negative frequencies.

Appendix C

The ZR Component

From Haney et al. (2012), the time-domain expression for the ZR correlation is given for times prior to the direct arrival \((|t| < r/c)\) as

$$\phi_{ZR}(r, t) = -\frac{2cR}{\pi r} \sum_{m=1}^{\infty} \text{Re}[y_m] U_{m-1} \left( \frac{ct}{r} \right), \quad (C1)$$

in which we again focus on times prior to the direct arrival because a causal propagating waveform will be equal to zero for \(|t| < r/c\). As in equation (B4), \( y_m \) is a measure of the distribution of energy propagation as a function of direction; however, it changes value depending whether ZZ or ZR is being considered. In the case of ZR, \( y_1 = 1/2 \) and \( y_m = 0 \) for \( m \neq 1 \) for an isotropic wavefield (Haney et al., 2012). From equation (C1), this means that for times prior to the direct arrival \((|t| < r/c)\),

$$\phi_{ZR}(r, t) = -\frac{cR}{\pi r}, \quad (C2)$$

which is a constant independent of time. Although \( \phi_{ZR}(r, t) \) is nonzero for \(|t| < r/c\), the time derivative of \( \phi_{ZR}(r, t) \) is equal to zero. Thus, \( d\phi_{ZR}/dt \) is the sum of a causal propagating waveform and a time-reversed version of a causal propagating waveform. For purposes of applying the Kramers–Kronig relations, we need to find how \( d\phi_{ZR}/dt \) is related to \( G_{ZR} \).

From equation (B1), we obtain

$$\phi_{ZR}(t) = 2[H[G_{ZR}(t)] - H[G_{ZR}(-t)]], \quad (C3)$$

because \( H^{-1} = -H \). By taking a time derivative, we arrive at an expression for \( d\phi_{ZR}/dt \):

$$\frac{d\phi_{ZR}}{dt} = 2 \left\{ \frac{dH[G_{ZR}(t)]}{dt} - \frac{dH[G_{ZR}(-t)]}{dt} \right\}. \quad (C4)$$

From equation (C4), \( dH[G_{ZR}] \)/\( dt \) must be a causal function and, as a result, the Kramers–Kronig relations can be applied to it.

Based on our Fourier domain convention, Hilbert transformation and differentiation are jointly represented in the Fourier domain as \(-|\omega|\). Therefore, in the Fourier domain, the causal function we are interested in is \(-|\omega|G_{ZR} \). The imaginary part of this function is obtained from equation (B3):

$$4\text{Im}(-|\omega|G_{ZR}) = -|\omega|\text{sgn}(\omega)\phi_{ZR} = -\omega\phi_{ZR}. \quad (C5)$$

From equation (1), we know that

$$\text{Im}(-|\omega|G_{ZR}) = \frac{R\omega}{4} J_1(|\omega| r/c). \quad (C6)$$

in which we have again set the spectral power in equation (1) to unity \([P(\omega) = 1]\). Based on the Kramers–Kronig relation from equation (A11), we find that

$$\text{Re}(-|\omega|G_{ZR}) = \frac{R}{4} H[|\omega| J_1(|\omega| r/c)]. \quad (C7)$$
We use the following identity (Erdelyi, 1954, p. 254, their equation 14 with \( \nu = 1 \)):

\[
H[xJ_1(|x|)] = -|x|Y_1(|x|), \tag{C8}
\]

in which \( Y_1 \) is the first-order Bessel function of the second kind, and obtain the following for the real part

\[
\text{Re}(G_{ZR}) = \frac{R}{4} Y_1(|a|/r/c). \tag{C9}
\]

From equations (C6) and (C9), we obtain the exact form for the \( ZR \) component of the Rayleigh-wave Green’s tensor:

\[
G_{ZR} = \frac{R}{4}[Y_1(|a|/r/c) - \text{sgn}(\omega)J_1(|a|/r/c)]. \tag{C10}
\]

As was the case for equation (B9), this is valid at all distance ranges including the near field.

The same considerations discussed previously for \( G_{ZZ} \) concerning normalization and sources and receivers at different depths applies to \( G_{ZR} \) as well. For positive frequencies, we find that

\[
G_{ZR} = -\frac{ir_2(z)r_1(h)}{8cU_1}H_1^{(1)}(\omega r/c). \tag{C11}
\]

The expression for negative frequencies follows from the procedure described in Appendix B. Last, due to the symmetries in equation (1) and taking into account the different source and receiver depths, the \( G_{RZ} \) component is given by

\[
G_{RZ} = \frac{ir_1(z)r_2(h)}{8cU_1}H_1^{(1)}(\omega r/c), \tag{C12}
\]

which gives the following relation between the two components: \( G_{RZ} = -\frac{r_1(z)r_2(h)}{r_2(z)r_1(h)}G_{ZR} \). When the source and receiver are at the same depth, these two components are opposite in sign.

Appendix D

The \( RR \) and \( TT \) Components

The \( RR \) and \( TT \) components in the Rayleigh-wave Green’s tensor follow from the \( RR \) and \( TT \) correlations in equation (1). The \( RR \) and \( TT \) correlations for Love waves in equation (2) have the same form as the Rayleigh correlations for \( TT \) and \( RR \), respectively. Thus, the Green’s tensor result for Rayleigh waves leads directly to the expression for Love waves. Note that although the \( TT \) component of the Rayleigh-wave Green’s tensor is zero in the far field, in the near field it can be nonzero. Similarly, the \( RR \) component of the Love-wave Green’s tensor is in general nonzero.

From Haney et al. (2012), the time-domain expression for the \( TT \) Rayleigh-wave correlation is given for times prior to the direct arrival \((|t| < r/c)\) as

\[
\phi_{TT}(r,t) = \frac{2cR^2}{\pi r^2} \sum_{m=1}^{\infty} \text{Re}[\gamma_m]U_{m-1}\left(\frac{ct}{r}\right). \tag{D1}
\]

As in equation (B4), \( \gamma_m \) is a measure of the distribution of energy propagation as a function of direction; however, it changes value depending whether \( ZZ \) or \( TT \) is being considered. In the case of \( TT \) for an isotropic wavefield, the only nonzero \( \gamma_m \) are \( \gamma_0 = 1/2 \) and \( \gamma_2 = -1/2 \) (Haney et al., 2012). From equation (D1), this means that for times prior to the direct arrival \((|t| < r/c)\),

\[
\phi_{TT}(r,t) = -\frac{2c^2R^2t}{\pi r^2} \tag{D2}
\]

which is a linear function of time. Although \( \phi_{TT}(r,t) \) is nonzero for \( |t| < r/c \), its second time derivative is equal to zero. Thus, \( d^2\phi_{TT}/dt^2 \) is the sum of a causal propagating waveform and a time-reversed version of a causal propagating waveform. For purposes of applying the Kramers–Kronig relations, we need to find how \( d^2\phi_{TT}/dt^2 \) is related to \( G_{TT} \).

From equation (B1), we obtain the following expression by taking two time derivatives:

\[
H \left[ \frac{d^2\phi_{TT}}{dt^2} \right] = -2 \left[ \frac{d^2G_{TT}(t)}{dt^2} - \frac{d^2G_{TT}(-t)}{dt^2} \right]. \tag{D3}
\]

From equation (D3), \( d^2G_{TT}/dt^2 \) must be a causal function and, as a result, the Kramers–Kronig relations can be applied to it. Because double differentiation is represented in the Fourier domain as \(-\omega^2\), the causal function we are interested in is \(-\omega^2G_{TT}\). The imaginary part of this function is obtained from equation (B3):

\[
4\text{Im}(-\omega^2G_{TT}) = -\omega^2\text{sgn}(\omega)\phi_{TT}. \tag{D4}
\]

From equation (1), we know that

\[
\text{Im}(-\omega^2G_{TT}) = -\omega^2\text{sgn}(\omega)\frac{R^2}{8}[J_0(\omega r/c) + J_2(\omega r/c)], \tag{D5}
\]

in which the spectral power in equation (1) has been set to unity: \( R^2(\omega) = 1 \). Based on the Kramers–Kronig relation in equation (A11), the associated real part is

\[
\text{Re}(-\omega^2G_{TT}) = -\frac{R^2}{8}H[\omega^2\text{sgn}(\omega)[J_0(\omega r/c) + J_2(\omega r/c)]]. \tag{D6}
\]

Because of the recurrence relation of Bessel functions of the first kind,

\[
\frac{2c}{\omega r}J_2(\omega r/c) = J_0(\omega r/c) + J_2(\omega r/c), \tag{D7}
\]

equation (D6) can be rewritten as
\[ \text{Re}(-\omega^2 G_{TT}) = -\frac{cR^2}{4r} H[\omega \text{sgn}(\omega) J_1(\omega r/c)] \]
\[ = -\frac{cR^2}{4r} H[\omega J_1(|\omega| r/c)], \quad \text{(D8)} \]
in which we have utilized the fact that the first-order Bessel function is odd. The Hilbert transform in equation (D8) is recognized to have the same form as in equation (C7). Thus, we use the same identity as before (Erdelyi, 1954, p. 254, equation 14 with \( \nu = 1 \)),
\[ H[xJ_1(|x|)] = -|x|Y_1(|x|), \quad \text{(D9)} \]
and obtain the following for the real part
\[ \text{Re}(-\omega^2 G_{TT}) = \frac{cR^2}{4r} |\omega| Y_1(|\omega| r/c). \quad \text{(D10)} \]
Because of the recurrence relation of Bessel functions of the second kind,
\[ \frac{2c}{|\omega|} Y_1(|\omega| r/c) = Y_0(|\omega| r/c) + Y_2(|\omega| r/c), \quad \text{(D11)} \]
we find that equation (D10) can be rewritten as
\[ \text{Re}(-\omega^2 G_{TT}) = \frac{R^2}{8} \omega^2 [Y_0(\omega r/c) + Y_2(\omega r/c)]. \quad \text{(D12)} \]
Because \( \omega^2 \) is a real number, the previous equation can be simplified to yield the following for the real part
\[ \text{Re}(G_{TT}) = -\frac{R^2}{8} [Y_0(\omega r/c) + Y_2(\omega r/c)]. \quad \text{(D13)} \]

From equations (D5) and (D13), we obtain the exact form for the \( TT \) component of the Rayleigh-wave Green’s tensor:
\[ G_{TT} = \frac{R^2}{8} \{-[Y_0(\omega r/c) + Y_2(\omega r/c)] \]
\[ + i \text{sgn}(\omega) [J_0(\omega r/c) + J_2(\omega r/c)]\}. \quad \text{(D14)} \]
The same considerations concerning normalization and sources and receivers at different depths applies to \( G_{TT} \) as well. For positive frequencies, we find that
\[ G_{TT} = \frac{i r_1(z) r_1(h)}{16c U_1} [H_0^{(1)}(\omega r/c) + H_2^{(1)}(\omega r/c)]. \quad \text{(D15)} \]
The expression for negative frequencies follows from the procedure described in Appendix B. In a similar fashion, the \( G_{RR} \) component can be shown to be equal to
\[ G_{RR} = \frac{i r_1(z) r_1(h)}{16c U_1} [H_0^{(1)}(\omega r/c) - H_2^{(1)}(\omega r/c)]. \quad \text{(D16)} \]
The forms of equations (D15) and (D16) are the same for the \( RR \) and \( TT \) components of the Love-wave Green’s tensor, respectively. The only modification is the substitution of the Love-wave eigenfunction \( l_1 \) for \( r_1 \) and the association of the various parameters \( (c, U, \text{and } I_1) \) with Love waves.

U.S. Geological Survey
Volcano Science Center
Alaska Volcano Observatory
4210 University Dr.
Anchorage, Alaska 99508
mhaney@usgs.gov
(M.M.H.)

Department of Geophysics
Graduate School of Science
Tohoku University
Aoba-ku, Sendai 980-8578, Japan
naka@zisin.gp.tohoku.ac.jp
(H.N.)

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