Multiple scattering formulation for 3D georadar problem

Partha S. Routh and Timothy C. Johnson, Dept. of Geosciences, Boise State University, Boise, ID*

Summary

In this paper we present a multiple scattering formulation to incorporate the effects of electrical conductivity in the attenuation of GPR responses. Multiple scattering is based on the integral representation of the electric field in the presence of scatters in a background medium. The propagating wavefield due to interaction of different length scale scatterers is solved as a scattering series solution of the Helmholtz equation for the electric field. We concentrate on the borehole attenuation tomography problem and therefore obtain the solution for sources inside the medium. Borehole GPR tomography is an useful tool for imaging temporal and spatial changes in bulk conductivity due to fluid flow in the medium. Due to the computation complexity of the full scattering matrix we parallelize the computation steps to obtain the solution in a realistic time. This paper focuses on the forward scattering solution as a multiple scattering problem.

Introduction

It is well recognized that the near surface environment is complicated due to the presence of heterogeneity at different length scales. The presence of multi-scale heterogeneities affects the wave propagation of electromagnetic waves in the medium. The information in the recorded data have contributions from different length scale scatterers. In a borehole radar problem when an electric wavefield impinges on a heterogeneous conductor it is scattered. If there are many scatterers in the medium the wavefield scattered from one conductor will induce more scattering from other conductors and the resultant wavefield will be a mixture of these scattered fields. This is the multiple scattering phenomena which has been widely studied in many branches of science (Lax, 1951; Cui et al., 1998; Snieder, 1999; Spetzler and Snieder, 2001). Traditional techniques to solve the borehole radar problem are based on ray theory. Ray theory is valid only if the wavelengths of waves are smaller than scale of heterogeneities (or scatterers) in the medium. That is, the widths of the Fresnel zones are smaller than the characteristic length scale of scatterers. Although ray theory provides a very fast solution, significant artifacts can result due to the inappropriate physics (Johnson et al., 2005). Johnson et. al (2005) shows that accounting for finite frequency propagation effects in the attenuation difference modeling using the first Fresnel volume provides a more accurate solution for bulk conductivities in a borehole radar problem.

Methodology

The frequency domain Helmholtz equation for the electric field $\mathbf{E} = \mathbf{E}(\omega, \mathbf{r})$ in a background 3D conductive medium is given by

$$\nabla^2 \mathbf{E}_0 + k^2 \mathbf{E}_0 = S(\omega)\delta(\mathbf{r} - \mathbf{r}_s) \tag{1}$$

where $\mathbf{E}_0$ is the background electric field and the propagation constant $k^2 = -i\omega\mu\sigma_0 + \omega^2\mu\epsilon$ with background conductivity σ_0 and $\mathbf{r}_s$ is the source location. Our goal in the attenuation problem is to determine the change in the electric field due to conductivity perturbation $\delta\sigma$. Neglecting the second order terms (i.e $\delta\sigma\delta\mathbf{E}$) in the perturbed equation we obtain the Helmholtz equation for the perturbed electric field $\delta\mathbf{E}(\mathbf{r})$, given by

$$\nabla^2(\delta\mathbf{E}) + k^2\delta\mathbf{E} = i\omega\mu\delta\sigma\mathbf{E} \tag{2}$$

Equations (2) and (1) have the same operator with different source terms in the right hand side, therefore we can use the Green's function approach to solve the problem. Both the background and the perturbed electric fields satisfy radiation boundary condition where the field goes to zero at infinite distance. Since eq(1) and eq(2) can be written componentwise, we can use the scalar 3D Green's function solution in whole space to solve the problem. The solution to the background field is the inner product of the Green's function with the source function i.e.,

$$\mathbf{E}_0(\omega, \mathbf{r}) = \int_V G(\mathbf{r}, \mathbf{r}')S(\omega)\delta(\mathbf{r}' - \mathbf{r}_s)dV'. \tag{3}$$

Similarly the solution to the perturbed electric field $\delta\mathbf{E}$ can be obtained using Green's function in eq(2)

$$\delta\mathbf{E}(\omega, \mathbf{r}) = i\omega\mu \int_V G(\mathbf{r}, \mathbf{r}')\delta\sigma(\mathbf{r}')\mathbf{E}(\mathbf{r}')dV'. \tag{4}$$

Equation (4) is the integral solution to the scattered field or the perturbed field. We note that the above integral vanishes in the region where the conductivity perturbation is zero. Therefore we are only required to know the total field in the region where $\delta\sigma \neq 0$. Equation (4) is commonly referred to Fredholm integral equation of second kind where the integrand is a function of the unknown total electric field $\mathbf{E}(\mathbf{r}') = \mathbf{E}_0(\mathbf{r}') + \delta\mathbf{E}(\mathbf{r}')$. The difficulty in solving the above equation is that the quantity of interest $\delta\mathbf{E}$ appears on both sides of the equation. Dropping the dependence on ω for notational convenience eq(4) can be written in two parts

$$\delta\mathbf{E}(\mathbf{r}) = i\omega\mu\int_V G(\mathbf{r},\mathbf{r}')\delta\sigma(\mathbf{r}')\mathbf{E}_0(\mathbf{r}')dV' \tag{5a}$$

$$+i\omega\mu\int_V G(\mathbf{r},\mathbf{r}')\delta\sigma(\mathbf{r}')\delta\mathbf{E}(\mathbf{r}')dV'. \tag{5b}$$

If we neglect the second integral in eq(5), it effectively amounts to replacing the total field $\mathbf{E}(\mathbf{r}')$ in eq(4) by the background field $\mathbf{E}_0(\mathbf{r}')$ commonly known as *Born approximation*. To proceed further we parameterize the medium into cells of small elemental volume such that each cell can be considered as a point scatterer. We consider an ensemble of P point scatterers in the medium that can be mathematically represented by

$$\delta\sigma(\mathbf{r}') = \sum_{j=1}^{P}\delta\sigma_j\delta(\mathbf{r}'-\mathbf{r}_j)v_j \tag{6}$$

where $\mathbf{r}_j$ is the location of the j^{th} scatterer in the medium occupying an elemental volume v_j. And $\delta\sigma_j$ is the strength of the j^{th} scatterer. Substituting eq(6) in eq(5) and using Dirac sampling theorem we obtain

$$\delta\mathbf{E}(\mathbf{r}) = i\omega\mu\sum_{j=1}^{P}G(\mathbf{r},\mathbf{r}_j)\delta\sigma_j v_j\mathbf{E}_0(\mathbf{r}_j) \tag{7a}$$

$$+i\omega\mu\sum_{j=1}^{P}G(\mathbf{r},\mathbf{r}_j)\delta\sigma_j v_j\delta\mathbf{E}(\mathbf{r}_j). \tag{7b}$$

Equation (7) is the representation of the scattered field in discretized form where the second summation depends on the quantity $\delta E(\mathbf{r}_j)$ we are trying to solve. We note that the second summation only contributes where the scattering strength $\delta\sigma_j$ is non-zero. If we take this term onto the left hand side we can write

$$\delta\mathbf{E}(\mathbf{r}) - i\omega\mu\sum_{j=1}^{P}G(\mathbf{r},\mathbf{r}_j)\delta\sigma_j v_j\delta\mathbf{E}(\mathbf{r}_j) \tag{8a}$$

$$= i\omega\mu\sum_{j=1}^{P}G(\mathbf{r},\mathbf{r}_j)\delta\sigma_j v_j\mathbf{E}_0(\mathbf{r}_j). \tag{8b}$$

Consider that we are interested in computing the scattered field at position $\mathbf{r}_k$. This can be easily achieved by replacing $\mathbf{r}=\mathbf{r}_k$ in eq(8). However in doing so we will encounter a situation when $k=j$ such that $\mathbf{r}_k=\mathbf{r}_j$ and the Green's function is singular (i.e. not defined). To avoid the singularity we drop the contribution of the summation in eq(8) when $\mathbf{r}_k=\mathbf{r}_j$. Physically this implies that a scatterer cannot scatter from itself which is a realistic situation. To compute the scattered field we discretize the whole medium into M cells and evaluate the perturbed electric field $\delta\mathbf{E}(\mathbf{r}_k)$ in the cell center of each elemental cell i.e. at point r_k. This is given by

$$\delta\mathbf{E}(\mathbf{r}_k) - i\omega\mu\sum_{j=1(j\neq k)}^{P}G(\mathbf{r}_k,\mathbf{r}_j)\delta\sigma_j v_j\delta\mathbf{E}(\mathbf{r}_j) = \tag{9a}$$

$$i\omega\mu\sum_{j=1(j\neq k)}^{P}G(\mathbf{r}_k,\mathbf{r}_j)\delta\sigma_j v_j\mathbf{E}_0(\mathbf{r}_j)\text{ for }k=1,\cdots,M. \tag{9b}$$

The scattering strength $\delta\sigma_j v_j$ can be written as a diagonal matrix C size $M\times M$ which we call the scattering strength matrix. The cells that do not contain scatterers (i.e. scattering strength is zero) the corresponding diagonal element for that cell index will be zero. This allows us to write the summation from $j=1,\cdots,M$ in eq(9). Equation (9) can be represented by

$$\sum_{j=1(j\neq k)}^{M}\left[1-i\omega\mu G(\mathbf{r}_k,\mathbf{r}_j)C_{jj}\right]\delta\mathbf{E}(\mathbf{r}_j) \tag{10a}$$

$$= i\omega\mu\sum_{j=1(j\neq k)}^{M}G(\mathbf{r}_k,\mathbf{r}_j)C_{jj}\mathbf{E}_0(\mathbf{r}_j) \tag{10b}$$

where the scattering strength matrix C is given by

$$C = \mathrm{diag}(\delta\sigma_1 v_1,\delta\sigma_2 v_2,\cdots,\delta\sigma_M v_m) \tag{11}$$

Note that only P of the diagonal elements in C will be non-zero. Equation (10) shows that the scattered field need to be computed everywhere in the medium not just at receiver locations. This is not surprising since the scattered solution at position $\mathbf{r}_k$ i.e. $\delta\mathbf{E}(\mathbf{r}_k)$ is a function of the scattered field within the scattering volume i.e. $\delta\mathbf{E}(\mathbf{r}_j)$, shown by eq(11). Therefore eq (10) can be written in a matrix form given by

$$(I-Q)\,\delta\mathbf{E}(\mathbf{r}) = \quad Q\mathbf{E}_0(\mathbf{r}) \tag{12}$$

The matrix operator Q in eq (12) is the scattering matrix operator. Q essentially summarizes the summation process in eq(10) and is the projection of the Green's function scaled by the scattering strength matrix C onto the scattered field $\delta\mathbf{E}$ on the right hand side and onto the background field $\mathbf{E}_0(\mathbf{r})$ in the left hand side of eq(12). The elements of scattering matrix $Q_{jk}=Q(\mathbf{r}_k,\mathbf{r}_j,\omega)$ are given by i.e.,

$$Q_{jk} = \quad -i\omega\mu G(\mathbf{r}_k,\mathbf{r}_j)C_{jj} \quad \text{for } k\neq j \tag{13a}$$

$$= \quad 0 \quad \text{for } k=j \tag{13b}$$

Equation (13) is similar to that obtained by Snieder (1999). If we invert the operator $(I-Q)$ in eq(12) we obtain an expression for the perturbed electric field in terms of the background field. This is commonly called the inverse scattering approach which is given by

$$\delta\mathbf{E}(\mathbf{r}) = \quad (I-Q)^{-1}\,Q\mathbf{E}_0(\mathbf{r}) \tag{14}$$

The scattering matrix Q contains Green's functions scaled by conductivity perturbations. Assuming that the conductivity perturbations are small such that $\|Q\| << 1$ the inverse operator $(I-Q)^{-1}$ can be expanded in a binomial series. This can be represented by

$$\delta\mathbf{E}(\mathbf{r}) = \left(I + Q + Q^2 + Q^3 + \cdots\right) Q\mathbf{E}_0(\mathbf{r}) \qquad (15)$$

Equation (15) represents the scattered field solution where the first term $Q\mathbf{E}_0(\mathbf{r})$ is the Born approximation. Thus the total field $\mathbf{E}(\mathbf{r})$ is the sum of the background and the scattered field given by

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) + \delta\mathbf{E}(\mathbf{r}) \qquad (16a)$$

$$\mathbf{E}(\mathbf{r}) = \left(I + Q + Q^2 + Q^3 + Q^4 + \cdots\right) \mathbf{E}_0(\mathbf{r}) \qquad (16b)$$

Equation (16) is the scattering series solution. To obtain the perturbation solution due to multiple scattering we begin with the background field and successively multiply the scattering matrix Q with the background field and add the result to the total contribution. The first application of Q onto the background field is the Born approximation and successive terms provide the contribution due to high order scattering.

Computation of multiple scattered field

The solution for the scattered field in eq(12) shows that the field needs to be computed in the entire domain. For a particular source position $\mathbf{r}_s$ and for a single frequency (ω) we solve for $\delta\mathbf{E}(\mathbf{r},\omega,\mathbf{r}_s)$ using eq(12). However we are only interested in the solution at the receiver positions. Therefore we only keep the solutions at the receiver positions obtained at a suite of frequencies. The frequency domain solution is then transformed into time domain using inverse Fourier transform. We repeat this for each source position. The main computation expense in this method is the calculation of the scattering matrix. We parallelize the blocks of the scattering matrix into different nodes to meet the memory requirement.

Synthetic Results

To examine the effect of multiple scattering on scatterers we consider a causal 90 degree phase shifted Ricker wavelet with 100 MHz as the dominant frequency. The time sampling interval is 0.33ns and the Nyquist frequency of 1.5 GHz. The velocity of the medium is 0.0948 m/ns. Figure 1 has four point scatterer in adjacent cells, and cell dimensions are 25cm in all direction. The characteristic scale length of the body is 0.5m and is comparable to the dominant length scale of the wavelet of 0.948m. As indicated by several authors (Cui et al., 1998; Snieder, 1999; Spetzler and Snieder, 2001) the finite frequency propagation effects become important when length scales of the scatterer are comparable to the length scale of the probing signal. The background conductivity is 0.001S/m and the scatterers have conductivity of 0.01 S/m. Figure 1 (a)-(d) shows snapshots of the wavefield at different times computed using eq(16) with 6^{th} order scattering terms. A particular amplitude value is chosen to generate these isosurface plots of the wavefield. Therefore it shows how this particular amplitude progresses in time as it propagates in the medium. Figure 1(c) shows the reflected wavefield from the scatterer. To examine the scattering effect we include two large scatterers in the medium shown in Fig. 2. The third and fourth panel in Figure 2 clearly indicates the scattered signal between the two scatterers. To examine the nature of scattered signal we place a receiver at (1.9,0.7,0.7). Figure 3 shows the background, total response and the scattered contribution to the signal. Figure 3(c) clearly indicate that the magnitudes of the scattered signal is order of magnitude smaller than the background signal as expected.

Conclusions

In this paper we present the forward scattering solution of the 3D Helmholtz equation for the electric field to model attenuation response in borehole GPR data. The problem is formulated as a full scattering series solution. Higher order scattering can become important when the strength of scatterers are large and the characteristic length scales become comparable to the dominant length scale of the input signal.

Acknowledgments

We thank Roel Snieder from Colorado School of Mines for very helpful discussions on scattering. Computations were performed on the geophysical computing system (NSF-EPSCoR grant # EPS0132626) at CGISS. Funding for this work was provided by the Inland Northwest Research Alliance (Ph.D.fellowship for TJ).

References

Cui, T. J., Wiesback, W., and Herschlein, A., 1998, Electromagnetic scattering by multiple three-dimensional scatterers buried under multilayered media-part i theory: IEEE Transaction on geoscience and remote sensing, **36**, no. 2, 526–534.

Johnson, T. C., Routh, P. S., and Knoll, M. D., 2005, Fresnel volume georadar attenuation differenc tomography: Geophy. J. Inter. (accepted).

Lax, M., 1951, Multiple scattering of waves: Reviews of modern physics, **23**, no. 4, 287–310.

Snieder, R., 1999, Imaging and averaging in complex media: Diffuse waves in complex media, Ed. J. P. Fouque, Kluwer, pages 405–454.

Spetzler, J., and Snieder, R., 2001, The effect of small-scale heterogeneity on the arrival times of waves: Geophy. J. Inter., **145**, 786–796.

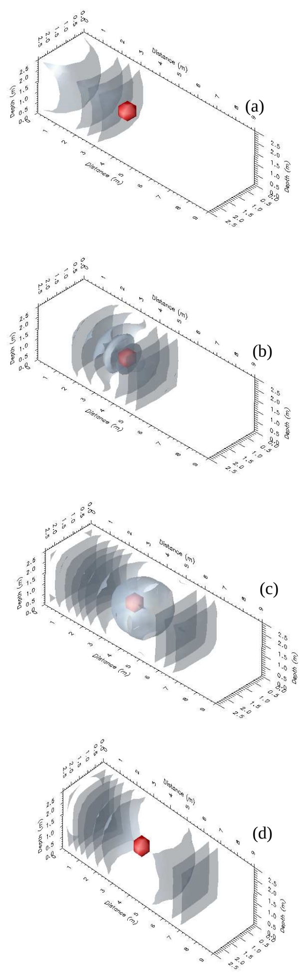

Fig. 1: Scattered field with a single large scatterer. This scatter is made of four point scatterers. The source is located at (0,1.25,1.25). This is an isosurface plot of a single amplitude of the propagating wavefield. (a)-(d) shows four time snap shots of the wavefield.

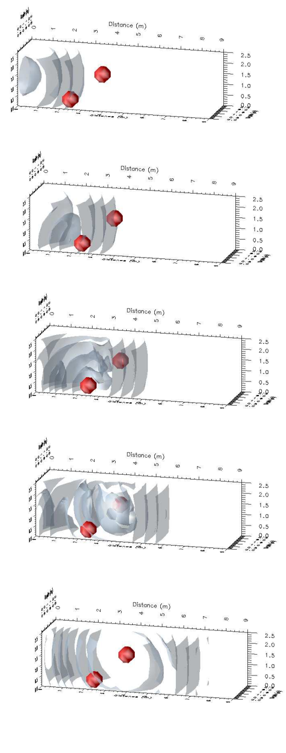

Fig. 2: Scattered field with two large scatterers. The source is located at (0,1.25,1.25). This is an isosurface plot of a single amplitude of the propagating wavefield. Notice the scattering in the third panel of this diagram

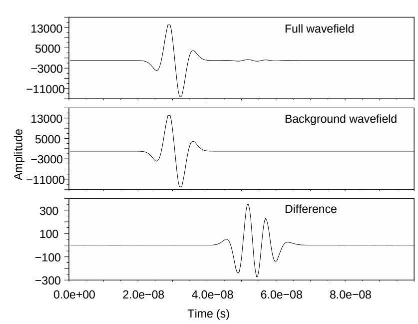

Fig. 3: Trace output with receiver positioned at (1.9,0.7,0.7). (a) Background trace without the scatterers, (b) total field trace with the scatterers and (c) The scattering contribution to the signal i.e. difference between total field and background trace.